# Complete Relational Description of Spin in a Quantum Background

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The standard description of the state of a spin in quantum mechanics presupposes externally fixed directions—a classical background. Can a spin be fully described instead in relation to other quantum mechanical systems? Poulin and Yard suggested twenty years ago that this may be achieved through the so-called incoherent average of the joint state of a fundamental spin and a reference spin with large angular momentum. However, this yields a classical spin in a probabilistic mixture. We revisit this idea and show that when the quantum reference system is augmented to *two* large spins, the encoding is complete. The standard quantum mechanical description of a spin is recovered in the limit of large quantum numbers of the reference.

## I. INTRODUCTION

The standard quantum mechanical quantization of a spin makes reference to a fixed external and classical frame of reference, which defines directions along which the spin is to be measured. This frame could be imagined to be physically realized, for instance, by the walls of the laboratory. More generally, this 'classical background' could be considered to be the (Euclidean) space, considered fixed once and for all.

The question arises: if all physical systems obey quantum mechanics, including spacetime, then, the general description of the physics of any quantum system—that of a spin in particular—should be given in relation to some other quantum system: in a 'quantum frame of reference' [1–4]. Motivated by the search of a quantum theory of gravity, and given that general relativity is independent of background, over the past decade much effort has gone into developing quantum information formalisms for quantum reference frames e.g [5–16]. For instance, this has lead to an understanding of the (quantum) relativity of entanglement [9, 15, 17] and localization of events [14, 16], depending on the choice of quantum reference frame. The description of a system relative to a QRF depends not just on properties of the system, but also of the frame: the algebra of relative observables depends on the choice of frame [18].

The examples found in the literature are typically concerned with a reference system that is a quantum system in the sense that it is in a spatial superposition. Remarkably, a concrete physical realization of a quantum reference frame for a system with quantized angular momentum has been missing. Here, we fill this gap by giving a fully relational description of a fundamental spin, with respect to a composite reference system consisting of two other systems with quantized angular momentum.

We had to deal with two difficulties, which may be reasons for this omission. First, while it seems intuitive that the state of spin could be described with respect to another physical system so long as it also points towards a direction, this only sufficient for a classical spin, not a quantum spin. One system of reference with angular momentum suffices to encode relationally that its joint state with the spin that is the system of interest is a (mostly) aligned or a (mostly) anti-aligned states. However, for a quantum mechanical description of a spin it is necessary to also capture complementarity: for this, a composite system of reference is needed, that has two angular momenta that do not commute. Second, as we shall see and as is usual, manipulations with recoupling theory (Clebsch-Gordan coefficients) quickly increase in complexity: working with three instead of two quantum systems with angular momentum (the spin and the two reference larger spins) yields a significantly more involved calculation.

The calculation presented in this work builds on an idea by Poulin and Yard circa 2007, where they consider the joint state of a spin-1/2 system with a second spin and remove the fiducial information of the coordinate axes by incoherently averaging the density matrix over all possible rotations [19]. However, taking the limit of large reference spin yields a mixed state which only partially encodes the initial state information of the fundamental spin (Section II). Here, in essence, we show that augmenting the quantum reference frame with a second large spin and following the same procedure yields a relational state that contains the full information about the pure state of the spin-1/2 system (Section III). In Section IV, we give several comments pertinent to the contemporary discussion on quantum reference frames and background independent formulations of quantum theory.

## II. QUBIT IN RELATION TO A LARGE SPIN

Poulin suggested the following strategy to extract a relational description of a quantum spin with respect to other quantum systems [19]: start from the standard

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quantum mechanical description of a spin which makes reference to classical externally defined directions, and 'remove' this background-dependence by group averaging over rotations in a certain way (known as an incoherent average for reasons that will be discussed in what follows).

Let us now illustrate Poulin's idea. Consider the spin-1/2 state

$$|\psi\rangle^{\mathcal{S}} = \alpha\,|{\uparrow}\rangle_z^{\mathcal{S}} + \beta\,|{\downarrow}\rangle_z^{\mathcal{S}}\,. \tag{1}$$

Model the $z$ as a second quantum system $\mathcal{G}$ with angular momentum $\hat{J}^{\mathcal{G}}$, which is in its highest magnetic moment state $|G,G\rangle_z^{\mathcal{G}}$. That is, $\hat{J}^{\mathcal{G}}\,|G,G\rangle_z^{\mathcal{G}} = \hat{M}_z^{\mathcal{G}}\,|G,G\rangle_z^{\mathcal{G}} = G\,|G,G\rangle_z^{\mathcal{G}}$, where $\hat{M}_z^{\mathcal{G}}$ is the magnetic moment of $\mathcal{G}$ in the $z$-direction. The state $|G,G\rangle_z^{\mathcal{G}}$ is chosen because it has semi-classical properties, in the sense that it saturates the Heisenberg uncertainty relation.

The spin system $\mathcal{S}$ and the reference system $\mathcal{G}$ do not interact with each other. In the total angular momentum eigenbasis, their joint state is

$$\begin{aligned}
|\psi\rangle^{\mathcal{S}\mathcal{G}} &= \left(\alpha\,|{\uparrow}\rangle_z^{\mathcal{S}} + \beta\,|{\downarrow}\rangle_z^{\mathcal{S}}\right)\otimes|G,G\rangle_z^{\mathcal{G}} \\
&= \alpha\,|G+\tfrac{1}{2},G+\tfrac{1}{2}\rangle_z^{\mathcal{S}\mathcal{G}} \\
&+ \frac{\beta}{\sqrt{2G+1}}\,|G+\tfrac{1}{2},G-\tfrac{1}{2}\rangle_z^{\mathcal{S}\mathcal{G}} \\
&+ \frac{\beta\sqrt{2G}}{\sqrt{2G+1}}\,|G-\tfrac{1}{2},G-\tfrac{1}{2}\rangle_z^{\mathcal{S}\mathcal{G}}
\end{aligned} \tag{2}$$

The dependence on the external $z$-axis direction can be removed by group averaging the state (written as a density matrix) over $SO(3)$

$$\mathcal{E}(\rho) = \int_{SO(3)} \mathrm{d}\mu(\Omega)\; R^{\mathcal{S}\mathcal{G}}(\Omega)\;\rho\; R^{\mathcal{S}\mathcal{G}}(\Omega)^{\dagger}, \tag{3}$$

where $\mathrm{d}\mu(\Omega)$ denotes the Haar measure of $SO(3)$. The above group averaging is called 'incoherent' because it acts with the same group element on both sides of the density matrix $\rho = |\psi\rangle\langle\psi|^{\mathcal{S}\mathcal{G}}$. It is in this sense a diagonal group averaging.

The representation $R^{\mathcal{S}\mathcal{G}}(\Omega)$ of the rotation group acts on the systems $\mathcal{S}$ and $\mathcal{G}$ as

$$R^{\mathcal{S}\mathcal{G}}(\Omega) = D^{1/2}(\Omega)\otimes D^{G}(\Omega), \tag{4}$$

where $D^{j}(\Omega)$ is the Wigner rotation matrix acting on the spin-$j$ representation space. The resulting state is

$$\begin{aligned}
\mathcal{E}(\rho) &= \left(|\alpha|^2 + \frac{|\beta|^2}{2G+1}\right)|G+\tfrac{1}{2}\rangle\langle G+\tfrac{1}{2}|\otimes\frac{\mathbb{1}_{2G+1}}{2G+1} \\
&+ \frac{|\beta|^2}{2G+1}|G-\tfrac{1}{2}\rangle\langle G-\tfrac{1}{2}|\otimes\frac{\mathbb{1}_{2G}}{2G}.
\end{aligned} \tag{5}$$

The sector of the magnetic component along $z$ is now maximally mixed, the state for the background-dependent observable $\hat{M}_z^{\mathcal{G}}$ is proportional to the identity. Therefore, this state contains no information on $\hat{M}_z^{\mathcal{G}}$.

Tracing out the magnetic moment sector, we get the background-independent state

$$\begin{aligned}
\tilde{\mathcal{E}}(\rho) =& |\alpha|^2|G+\tfrac{1}{2}\rangle\langle G+\tfrac{1}{2}| \\
&+ \frac{|\beta|^2}{2G+1}|G+\tfrac{1}{2}\rangle\langle G+\tfrac{1}{2}| \\
&+ \frac{2G|\beta|^2}{2G+1}|G-\tfrac{1}{2}\rangle\langle G-\tfrac{1}{2}|
\end{aligned} \tag{6}$$

When $G \gg 1$ the second term is negligible. The amplitudes $|\alpha|$ and $|\beta|$ can be found by measuring the joint total angular momentum $j^{\mathcal{S}\mathcal{G}}$. When $G \gg 1$, the state is approximated by

$$\rho_{j^{\mathcal{S}\mathcal{G}}} = |\alpha|^2|G+\tfrac{1}{2}\rangle\langle G+\tfrac{1}{2}| + |\beta|^2|G-\tfrac{1}{2}\rangle\langle G-\tfrac{1}{2}|, \tag{7}$$

which is the probabilistic mixture of a classical spin with $|\alpha|^2$ probability up and $|\beta|$ probability down.

This can be seen differently as follows. If instead of the pure state (1), the point of departure was the mixed state

$$\rho^{\mathcal{S}} = |\alpha|^2|\uparrow\rangle\langle\uparrow|^{\mathcal{S}} + |\beta|^2|\downarrow\rangle\langle\downarrow|^{\mathcal{S}}, \tag{8}$$

the joint state of $\mathcal{S}$ and $\mathcal{G}$ becomes

$$\begin{aligned}
\rho^{\mathcal{S}\mathcal{G}} =& |\alpha|^2|\uparrow\rangle\langle\uparrow|^{\mathcal{S}}|G,G\rangle\langle G,G|^{\mathcal{G}} \\
&+|\beta|^2|\downarrow\rangle\langle\downarrow|^{\mathcal{S}}|G,G\rangle\langle G,G|^{\mathcal{G}}
\end{aligned} \tag{9}$$

In the total angular momentum eigenbasis, it reads

$$\begin{aligned}
\rho^{\mathcal{S}\mathcal{G}} =& |\alpha|^2|G+\tfrac{1}{2},G+\tfrac{1}{2}\rangle\langle G+\tfrac{1}{2},G+\tfrac{1}{2}|^{\mathcal{S}\mathcal{G}} \\
&+\frac{|\beta|^2}{2G+1}|G+\tfrac{1}{2},G-\tfrac{1}{2}\rangle\langle G+\tfrac{1}{2},G-\tfrac{1}{2}|^{\mathcal{S}\mathcal{G}} \\
&+\frac{2G|\beta|^2}{2G+1}|G-\tfrac{1}{2},G-\tfrac{1}{2}\rangle\langle G-\tfrac{1}{2},G-\tfrac{1}{2}|^{\mathcal{S}\mathcal{G}}
\end{aligned} \tag{10}$$

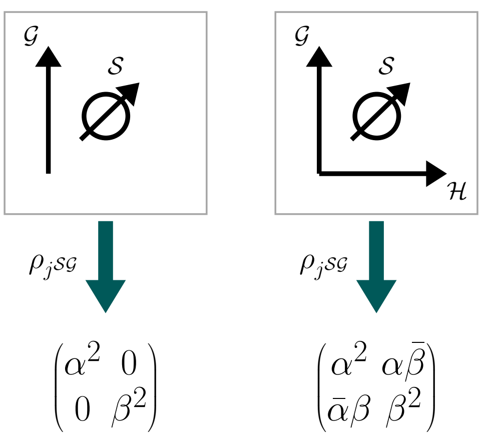


Figure 1. Illustrations of the physical situations under investigation. A spin-1/2 system $\mathcal{S}$ alongside a large spin system $\mathcal{G}$ would produce a background independent state corresponding to a mixture (left, Sec. II), whereas adding another large spin system $\mathcal{H}$ would result in a background independent state corresponding to $\mathcal{S}$'s original state relative to the now-absent background (right, Sec. III).

Group averaging and tracing out the magnetic component sector, this yields again (7).

Therefore, *a single large-spin reference system allows to encode only the magnitude of the qubit amplitudes, but not their relative phase.* This cannot distinguish, whether the original system of interest was in a quantum superposition state or a classical probabilistic mixture.

In Poulin's work, it was speculated that this 'classicalization' of the spin may have to do with using the incoherent average (3), and considering the 'classical limit' of large quantum number $G \gg 1$, resulting to some sort of effective 'measurement' of the spin. In the next Section, we will see that this is not the case: a pure spin state will be arrived at if two reference quantum systems are used.

## III. QUBIT IN RELATION TO *TWO* LARGE SPINS

We now augment the reference system with a second spin system $\mathcal{H}$ in a configuration orthogonal to $\mathcal{G}$. That is, the point of departure is the tensor product joint state of a spin-1/2 with two gyroscopes $\mathcal{G}$ and $\mathcal{H}$

$$|\psi\rangle^{\mathcal{SGH}} = \Big(\alpha\,|{\uparrow}\rangle_z^{\mathcal{S}} + \beta\,|{\downarrow}\rangle_z^{\mathcal{S}}\Big) \otimes |G,G\rangle_z^{\mathcal{G}} \otimes |H,H\rangle_x^{\mathcal{H}}\,. \quad (11)$$

Here, $|G,G\rangle_z^{\mathcal{G}}$ and $|H,H\rangle_x^{\mathcal{H}}$ correspond to orthogonal directions in the usual sense that the background Cartesian directions $x$ and $z$ are orthogonal.

Using the same logic presented in the previous section, we will find the state of $\mathcal{S}$ relative to the reference frame $\mathcal{GH}$ by applying the incoherent average in order to remove the dependence on the implicit fiducial $xyz$ system of axis. As we will see, adding a second reference large spin suffices to encode relationally the information that was missing from the state $\rho_{j^{\mathcal{SG}}}$ given in (7): the complex phase information of $\mathcal{S}$'s state.

Because we have two reference systems, $\mathcal{G}$ and $\mathcal{H}$, to use recoupling theory we need to make a choice on whether to first couple the spin system $\mathcal{S}$ to $\mathcal{G}$, or first to $\mathcal{H}$. That is, on whether to first consider the recoupling basis over angular momenta $j^{\mathcal{SG}}$ or $j^{\mathcal{SH}}$. Below, we will first couple $\mathcal{S}$ to $\mathcal{G}$, and then the resulting $\mathcal{SG}$ system to $\mathcal{H}$. As we comment later on, with this calculation at hand, the results for when $\mathcal{S}$ is first coupled to $\mathcal{H}$ can be also extracted by inspection. The calculation that follows is much more involved than the case of one reference large spin. In the remaining of this Section, we sketch the main steps and comment on the points of conceptual importance. The detailed calculation is given in Appendix A.

### A. Basis Change

We start with changing basis to the eigenbasis that has eigenvalues the angular momentum $j^{\mathcal{SG}}$ of the joint system of $\mathcal{S}$ and $\mathcal{G}$. This is a change of basis of the form

$$\begin{aligned}\big\{|j^{\mathcal{S}},m^{\mathcal{S}}\rangle\big\} \otimes \big\{|j^{\mathcal{G}},m^{\mathcal{G}}\rangle\big\} \otimes \big\{|j^{\mathcal{H}},m^{\mathcal{H}}\rangle\big\} \\ \to \big\{|j^{\mathcal{SG}},m^{\mathcal{SG}},j^{\mathcal{S}},j^{\mathcal{G}}\rangle\big\} \otimes \big\{|j^{\mathcal{H}},m^{\mathcal{H}}\rangle\big\}\,.\end{aligned} \quad (12)$$

The eigenvalues $j^{\mathcal{S}} = S$ and $j^{\mathcal{G}} = G$ are constant, they are hereafter omitted to simplify notation. That is, we write

$$\big\{|j^{\mathcal{SG}},m^{\mathcal{SG}}\rangle\big\} \otimes \big\{|j^{\mathcal{H}},m^{\mathcal{H}}\rangle\big\}\,. \quad (13)$$

On this basis, the state $|\psi\rangle^{\mathcal{SGH}}$ is given by

$$\begin{aligned}|\psi\rangle^{\mathcal{SGH}} = \Big(&\alpha\,|G+\tfrac{1}{2},G+\tfrac{1}{2}\rangle \\ &+\frac{\beta}{\sqrt{2G+1}}\,|G+\tfrac{1}{2},G-\tfrac{1}{2}\rangle \\ &+\frac{\beta\sqrt{2G}}{\sqrt{2G+1}}\,|G-\tfrac{1}{2},G-\tfrac{1}{2}\rangle\Big) \otimes |H,H\rangle_x^{\mathcal{H}}\,.\end{aligned} \quad (14)$$

The prefactors of each term are the Clebsch-Gordan coefficients corresponding to joining a spin-1/2 with a spin $G$.

Now, we change basis again to the eigenbasis that has eigenvalues the angular momentum $j^{\mathcal{SGH}}$ of the joint system of $\mathcal{S}$, $\mathcal{G}$ and $\mathcal{H}$. This is of the form

$$\big\{|j^{\mathcal{SG}},m^{\mathcal{SG}}\rangle\big\} \otimes \big\{|j^{\mathcal{H}},m^{\mathcal{H}}\rangle\big\} \to \big\{|j^{\mathcal{SGH}},m^{\mathcal{SGH}},j^{\mathcal{SG}}\rangle\big\}\,, \quad (15)$$

where, as before, we omit the constant $j^{\mathcal{H}} = H$. To implement this basis change we have to rewrite the state $|H,H\rangle_x^{\mathcal{H}}$ in the $z$-basis:

$$|H,H\rangle_x^{\mathcal{H}} = \frac{1}{2^H}\sum_{h=-H}^{H}\sqrt{\binom{2H}{H+h}}\,|H,h\rangle_z^{\mathcal{H}}\,. \quad (16)$$

Replacing this into (14) and defining the Clebsch-Gordan coefficients

$$C^{j^{\mathcal{SGH}}m^{\mathcal{SGH}}}_{j^{\mathcal{SG}}m^{\mathcal{SG}}m^{\mathcal{H}}} := \langle j^{\mathcal{SGH}}m^{\mathcal{SGH}}j^{\mathcal{SG}}|j^{\mathcal{SG}}m^{\mathcal{SG}}j^{\mathcal{H}}m^{\mathcal{H}}\rangle\,, \quad (17)$$

the state takes the form

$$\begin{aligned}|\psi\rangle^{\mathcal{SGH}} = &\frac{1}{2^H}\sum_{h=-H}^{H}\sum_{J=G-1/2+h}^{G+1/2+H}\sqrt{\binom{2H}{H+h}} \\ &\Bigg[\alpha\, C^{J,G+1/2+h}_{G+1/2,G+1/2,h}\,|J,G+\tfrac{1}{2}+h,G+\tfrac{1}{2}\rangle \\ &+\frac{\beta}{\sqrt{2G+1}}C^{J,G-1/2+h}_{G+1/2,G-1/2,h}\,|J,G-\tfrac{1}{2}+h,G+\tfrac{1}{2}\rangle \\ &+\frac{\sqrt{2G}\,\beta}{\sqrt{2G+1}}C^{J,G-1/2+h}_{G-1/2,G-1/2,h}\,|J,G-\tfrac{1}{2}+h,G-\tfrac{1}{2}\rangle\Bigg],\end{aligned} \quad (18)$$

see Appendix A for intermediate steps. We use the convention that $C^{j^{\mathcal{SGH}}m^{\mathcal{SGH}}}_{j^{\mathcal{SG}}m^{\mathcal{SG}}m^{\mathcal{H}}} = 0$ when $j^{\mathcal{SGH}} < m^{\mathcal{SGH}}$ or $j^{\mathcal{SGH}} > j^{\mathcal{SG}} + H$.

### B. Group Averaging

To obtain the relational state of the system which makes no reference to fixed external directions $xyz$, we now calculate the incoherent group average of the density matrix $\rho^{\mathcal{SGH}} = |\psi\rangle\langle\psi|^{\mathcal{SGH}}$ corresponding to the state (18) (given explicitly in Appendix A). In analogy to (3), the incoherent group average operation is defined as

$$\mathcal{E}(\rho^{\mathcal{SGH}}) = \int_{SO(3)} \mathrm{d}\mu(\Omega)\ R^{\mathcal{SGH}}(\Omega)\ \rho^{\mathcal{SGH}}\ R^{\mathcal{SGH}}(\Omega)^{\dagger}. \tag{19}$$

To simplify notation, below we write directly the state $\tilde{\mathcal{E}}(\rho^{\mathcal{SGH}})$, which is the state after tracing out the identity on magnetic moment sector (see (5) and (6) in the previous Section). We get

$$\begin{aligned}\tilde{\mathcal{E}}(\rho^{\mathcal{SGH}}) = \sum_{J=G-1/2-H}^{G+1/2+H} \Bigg[ & \\ & \frac{|\alpha|^2}{2}\, S^{(1)}_{JGH} |J, G+\tfrac{1}{2}\rangle\langle J, G+\tfrac{1}{2}| \\ & + \frac{\sqrt{2G}\alpha\bar{\beta}}{\sqrt{2G+1}} S^{(2)}_{JGH} |J, G+\tfrac{1}{2}\rangle\langle J, G-\tfrac{1}{2}| \\ & + \frac{G\,|\beta|^2}{2G+1}\, S^{(3)}_{JGH} |J, G-\tfrac{1}{2}\rangle\langle J, G-\tfrac{1}{2}| \\ & + \frac{\sqrt{2G}\,|\beta|^2}{2G+1}\, S^{(4)}_{JGH} |J, G+\tfrac{1}{2}\rangle\langle J, G-\tfrac{1}{2}| \\ & + \frac{\alpha\bar{\beta}}{\sqrt{2G+1}} S^{(5)}_{JGH} |J, G+\tfrac{1}{2}\rangle\langle J, G+\tfrac{1}{2}| \\ & + \frac{|\beta|^2}{2(2G+1)}\, S^{(6)}_{JGH} |J, G+\tfrac{1}{2}\rangle\langle J, G+\tfrac{1}{2}| \Bigg] + h.c.\end{aligned} \tag{20}$$

Here, the $S^{(i)}_{JGH}$ are real numbers such that $-1 \leq S^{(i)}_{JGH} \leq 1$. They depend on $j^{\mathcal{SGH}} = J$, $j^{\mathcal{G}} = G$ and $j^{\mathcal{H}} = H$ and are defined explicitly in Appendix A.

### C. Large Gyroscope Limit

Now, we want to ask whether the above state has a reasonable classical limit, in the following sense. When we take the angular momentum $G$ and $H$ to be large, the corresponding coherent states become orthogonal. In this limit, we then expect to recover the usual state of a qubit that we started from, with $G$ and $H$ behaving as two orthogonal classical axis.

Verifying that that this is the case requires a numerical investigation, due to the complicated formulas arising from the CG coefficients, and is given in the Appendix refapp:LargeLimit. The main points are as follows. The coefficients $S^{i}_{JGH}$ are all bounded from below by -1 and from above by 1. Then, by inspection of (20), when $G \gg 1$, the matrix elements corresponding to $S^{4}_{JGH}$, $S^{5}_{JGH}$ and $S^{6}_{JGH}$ will be negligible due to the pre-factors that fall as $1/\sqrt{G}$, $1/\sqrt{G}$ and $1/G$ respectively. The matrix elements corresponding to $S^{1}_{JGH}$, $S^{2}_{JGH}$ and $S^{3}_{JGH}$ have pre-factors that fast become constant when $G \gg 1$. Therefore, these three only are not negligible. A technical point is that in order to carry out the calculation, their limit for $G \to \infty$ must exist. We give numerical evidence for this (see Appendix), which strongly implies that this is the case. Then, the approximate state $\tilde{\mathcal{E}}^{C}(\rho^{\mathcal{SGH}})$ becomes that of the qubit:

$$\begin{aligned}\tilde{\mathcal{E}}^{C}(\rho^{\mathcal{SGH}}) &= \rho_{j^{\mathcal{SGH}}} \otimes \rho_{j^{\mathcal{SG}}} \\ &= \rho_{j^{\mathcal{SGH}}} \otimes |\psi_{j^{\mathcal{SG}}}\rangle\langle\psi_{j^{\mathcal{SG}}}|\,, \\ |\psi_{j^{\mathcal{SG}}}\rangle &= \alpha\,|G+1/2\rangle + \beta\,|G-1/2\rangle\,.\end{aligned} \tag{21}$$

This is our main result. We have shown that the choice of the composite quantum reference frame system consisting of $\mathcal{G}$ and $\mathcal{H}$, suffices to fully reconstruct the state of a qubit in the limit of large angular momentum quantum numbers $G$ and $H$. the reason the left density matrix is classical is because of the incoherent averaging (which is incoherent over the entire system, it block diagonalizes/averages over the entire system). what happens to subsystems is non trivial, and as we see here it did not destroy the coherences of the SG subsystem because.

The letter $G$ remains here not as a variable, but only as part of the conventional label for the two different eigenstates of the operator $j^{\mathcal{SG}}$. The matrix $\rho_{j^{\mathcal{SGH}}}$ is a constant, numeric and diagonal density matrix that carries information about the gyroscopes only. It is defined in Appendix A and depicted in Fig. 2. The classical reference frame limit corresponding to a total angular momentum of $\sqrt{2}G$ — the length of the vector sum of the two orthogonal gyroscopes $\mathcal{G}$ and $\mathcal{H}$ — is approached as $G$ grows.

This completes the calculation. Several comments are in order.

## IV. COMMENTS

### A. No 'effective measurement' or collapse

there is no interaction so it could not be measurement. when we do the one axis thing we get something that looks like a projective measurement was done on one axis, explaining why we got the probabilistic mixture. what is actually happening is that the quantum system used as a reference did not have the 'resolution' to capture the information about the coherences because we needed it to have a non commuting observable in some other direction. having just one thing is like having cylindrical symmetry. this was actually happening for the total system angular momentum.

The protocol introduced in Sec. II does not constitute a measurement in the Von Neumann sense, as the different spin systems are assumed to be non-interacting. Rather than measuring a spin-1/2 particle, the large-spin particles merely allow us to reformulate its state in a

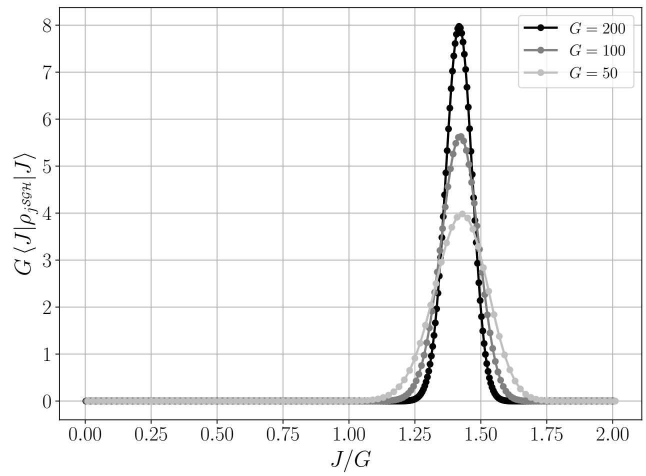


Figure 2. Total angular momentum probability densities of the $\mathcal{SGH}$ system for $G = 50, 100, 200$, as encoded by the density matrix $\rho_{j\mathcal{SGH}}$ in (21). The peak occurs at $J \approx \sqrt{2}G$, consistent with the sum of two orthogonal vectors of equal lengths. Convergence to a delta function can be seen, indicating the classical limit for the reference system $\mathcal{GH}$ (while $\mathcal{S}$ remains quantum).

background-independent basis labeled by invariant quantum numbers. To extract information about $\mathcal{S}$ from this background-independent description, one would have to perform measurements on the final density matrix.

The final density matrix that emerges in the single-gyroscope case in Sec. II is diagonal. It would have also been obtained if one were to perform a projective measurement of the spin in the direction of $\mathcal{G}$'s state $\mathcal{S}$ (see also the discussion towards the end of Section II). However, interpreting the averaging process and $G \to \infty$ limit as some appearance of measurement or classicality [19] is false, as the two-gyroscope case shows. Indeed, the final density matrix there is not diagonal and corresponds to the general pure state of a spin-1/2 system, as that is part of the relational information contained in the total system (the other part being the distribution of $j^{\mathcal{SGH}}$). The reason for the final density matrix being diagonal in the single-gyroscope case is the absence of an orientation perpendicular to $\mathcal{G}$. This causes the relational information about $\mathcal{S}$ to be the (probabilities of) spin values in the direction of $\mathcal{G}$ only. The corresponding measurement scenario would be one where only measurements of a spin in one fixed direction are possible, and thus information about the complex phases of $\alpha$ and $\beta$ cannot be obtained.

### B. Coherent vs incoherent average

A well known gauge averaging procedure is the coherent group averaging, whereby a vector of a Hilbert space $|\psi\rangle$ is projected on its invariant component, that is

$$|\psi\rangle \mapsto \int_{\mathcal{G}} U(g)\,|\psi\rangle\ \mathrm{d}g. \tag{22}$$

Here $\mathcal{G}$ denotes an associated symmetry group and $\mathrm{d}g$ is the Haar measure of the same group. Turning this into an operation on density matrices one has

$$\Big(\int_{\mathcal{G}} U(g)\,|\psi\rangle\ \mathrm{d}g\Big)\Big(\int_{\mathcal{G}} \langle\psi|\,U^\dagger(g')\ \mathrm{d}g'\Big). \tag{23}$$

This is a different procedure than the incoherent average,

$$\rho = |\psi\rangle\langle\psi| \mapsto \int_{\mathcal{G}} U(g)|\psi\rangle\langle\psi|U^\dagger(g)\,\mathrm{d}g, \tag{24}$$

which involves a single 'diagonal' group integration over the density matrix.

The incoherent averaging erases the directional information (the magnetic component) and thus achieves a state with expectation values independent of a reference background. However, it does not fix the total angular momentum $j^{\mathcal{SGH}}$, implying that a reference background exists but is indefinite. The incoherent averaging on the other hand forces the nonexistence of a reference background and thus treats the angular momentum of the entire system as a gauge choice. It then fixes its arbitrary value by projecting the state onto the zero angular momentum subspace. While it was not treated in this work, it is likely to yield the same state for $\mathcal{SG}$.

An interesting way to understand the difference in the light of this work is as the difference between restricting to rotationally invariant pure *states* and restricting to rotationally invariant *properties of possible states*. Rotationally invariant pure states of angular momentum are only states with $j^{\mathcal{SGH}} = 0$, and restricting to them corresponds to the coherent averaging. On the other hand, some properties of a state might be invariant even if the states itself is not: the total angular momentum of any subsystem is a rotationally invariant property of any state. The incoherent averaging keeps the information regarding such invariant properties. In the two-gyroscope case, this information is encoded in $\rho_{j\mathcal{SGH}}$.

Interestingly, in the above analysis the total angular momentum of the entire system initially depended on the total angular momenta of the gyroscopes $G, H$. By taking the limit $G = H \to \infty$ this dependence of the relative state on them disappeared, reflecting the fact that as long that one knows that the gyroscopes are of equal spin that is much larger than that of $\mathcal{S}$, their precise value does not affect the system's observables. Taking the limit therefore eliminated the dependence on $G, H$ that remained after taking the incoherent average.

For completeness, one could define a background-independent and complete set of observables to describe $\mathcal{S}$ as the Pauli spin operators constructed from $j^{\mathcal{SG}} =$

$G+1/2$ and $j^{\mathcal{SG}} = G-1/2$:

$$\Sigma_1 = |G+1/2\rangle\langle G-1/2| + |G-1/2\rangle\langle G+1/2| \quad (25)$$
$$\Sigma_2 = -i|G+1/2\rangle\langle G-1/2| + i|G-1/2\rangle\langle G+1/2|$$
$$\Sigma_3 = |G+1/2\rangle\langle G+1/2| + |G-1/2\rangle\langle G-1/2|.$$

Note that the observables $\Sigma_1$, $\Sigma_2$ and $\Sigma_3$ are defined without making reference to $xyz$ axis, only the eigenbasis of the total angular momentum of $\mathcal{SG}$ is needed.

### C. Relation to reference frame quantization

The procedure we have implemented is an instance of reference frame quantization [20]. Given a symmetry group $G$, a system $\mathcal{H}_S$ and a reference system $\mathcal{H}_R$ transforming under $U_S(g)$ and $U_R(g)$ respectively the aim is to find a map from the system $S$ onto the $G$-invariant subsystem of $\mathcal{H}_S \otimes \mathcal{H}_R$. Explicitly this is finding maps:

$$\rho_S \mapsto \rho_{\text{inv}}, \quad (26)$$
$$E_S \mapsto E_{\text{inv}},$$

where $\rho_{\text{inv}} = \mathcal{G}(\rho_{\text{inv}})$ and $E_{\text{inv}} = \mathcal{G}(E_{\text{inv}})$ are invariant operators on $\mathcal{H}_S \otimes \mathcal{H}_R$ and $\mathcal{G}(\cdot) = \int_G (U_S(g) \otimes U_R(g))(\cdot)(U_S^\dagger(g) \otimes U_R^\dagger(g))\, \mathrm{d}g$.

Moreover we are interested in the case where $\mathrm{Tr}(\rho_S E_S) = \mathrm{Tr}(\rho_{\text{inv}} E_{\text{inv}})$ (possibly in some limit), which gives an exact encoding of $\mathcal{H}_S$ in the G-invariant subsystems of $\mathcal{H}_S \otimes \mathcal{H}_R$.

In [20] a general procedure for implementing an exact encoding is proposed (for compact groups $G$), which in the case of $G = \mathrm{SO}(3)$ differs from the one presented in this work.

The reference system is taken to be $\mathcal{H}_R \simeq L^2(G)$ acted on by the left regular representation: $U_R(h)\,|g\rangle = |hg\rangle$ where $\langle g|h\rangle = \delta_{g,h}$ for $g, h \in G$. The encoding is then:

$$\rho_S \mapsto \rho_{\text{inv}} = \frac{1}{N}\int_G U_S(g)\rho_S U_S^\dagger(g) \otimes |g\rangle\langle g|_R \,\mathrm{d}g, \quad (27)$$
$$E_S \mapsto E_{\text{inv}} = \int_G U_S(g)\rho_S U_S^\dagger(g) \otimes |g\rangle\langle g|_R \,\mathrm{d}g,$$

where $\mathrm{d}g$ is the Haar measure, and $N$ is some normalisation constant $N\mathbb{1} = \int |g\rangle\langle g|\, \mathrm{d}g$ (for a fully mathematically rigorous treatment of this integral see [7]).

This is a similar procedure to the one used in this work, except that we make use of a reference frame which is not isomorphic to $L^2(\mathrm{SO}(3))$. We provide a precise group representation theoretic characterisation of the $\mathcal{GH}$ reference system used in this work and contrast it to the $L^2(\mathrm{SO}(3))$ reference frame in Appendix B 2.

could not have simply used the results of 25 here because of this difference. here we do something more quantitative, more physically interesting. our reference frames are not isomorphic to L2(g) so we are not using the same reference frames. it is not clear how one would build this kind of reference frames using spins.

### D. Heuristic discussion for qudits

We have established that a pair of gyroscopes $\mathcal{G}, \mathcal{H}$ can fully resolve a single qubit state in the classical limit $G, H \to \infty$. One may wonder whether this result extends to qudits, which carry spin $\frac{d-1}{2}$ representations of SU(2). As shown in Appendix B 3 for any coherent state system $\{|\psi(g)\rangle\,|g \in G\}$ where $|\psi(g)\rangle \not\sim |\psi(h)\rangle$ for $h \neq g$, in the limit $\langle\psi(h)|\psi(g)\rangle \to \delta_{g,h}$ one has an ideal encoding: $\mathrm{Tr}(\rho_{\text{inv}} E_{\text{inv}}) \to \mathrm{Tr}(\rho_S E_S)$.

Hence any dimensional system transforming under SO(3) can be resolved by a pair of gyroscopes in the classical limit. The general proof in the Appendix is based on a more abstract characterization of the quantum reference frame (in terms of a system of group coherent states) and does not provide us with the relation between physical parameters characterizing the QRF (such as the total spin of the constituent parts of the gyroscope) and the encoded system.

### E. Emergence of standard quantum mechanics

Standard quantum mechanics is defined relative to an external classical reference frame, which may consist of a set of rods and clocks accessible to an experimenter.

We have shown that, in the case of a reference frame for direction, one can model the frame quantum mechanically and recover the standard quantum description of a spin-1/2 system.

Although each gyroscope is treated fully quantum mechanically throughout, the limit $G, H \to \infty$ makes spin coherent states[1] with distinct orientations effectively orthogonal, so that the reference systems have sharply distinguishable directions. This is reflected in the factorization $\mathcal{E}^{\lim}(\rho^{\mathcal{SGH}}) = \rho_{j^{\mathcal{SGH}}} \otimes \rho_{j^{\mathcal{SG}}}$, where $\rho_{j^{\mathcal{SG}}}$ approaches the pure state of $\mathcal{S}$ while $\rho_{j^{\mathcal{SGH}}}$ becomes sharply peaked around the classical value $J \approx \sqrt{2}G$ expected from adding two orthogonal vectors of equal length. In this sense, the emergence of the standard description is not tied to any measurement-like collapse, but to the availability of a sufficiently informative reference configuration whose ability to distinguish directions improves with an increase in the total spin magnitudes $G, H$.

## V. CONCLUSION

We showed how a spin-1/2 can be fully described in relation to another quantum system. The procedure followed was as follows. The point of departure is to write in the standard quantum mechanical formalism the joint

[1] States with the magnegtic component in some direction equal to the total angular momentum

tensor product state of a qubit $\mathcal{S}$ and two other quantum mechanical spins $\mathcal{G}$ and $\mathcal{H}$ that have arbitrary angular momentum. This presupposes external classical fixed directions, that define e.g. up or down as aligned or anti-aligned with the z-axis. Then, we applied the incoherent group average over $SO(3)$. This effectively averages over rotations and removes any reference to external fixed axis. Therefore, the directional information that remains is whether $\mathcal{S}$ is aligned or anti-aligned with one of the other two quantum mechanical spins. We have shown how this information can be extracted, giving an explicit example of how to fully describe a qubit in relation to a composite reference quantum system—a quantum refrence frame. We demonstrated that the resulting state of the qubit coincides in the limit of large quantum numbers for the quantum reference with the standard quantum mechanical state of a qubit.

Future directions suggested by our analysis are (i) extend the construction from a qubit to a general qudit system by taking $\mathcal{S}$ to carry the $d$–dimensional irreducible representation of SU(2) and repeating the same two-gyroscope twirling and large-spin limit. We expect two independent directional references to remain sufficient for reconstructing a generic qudit state, with the recovered state living on the appropriate $d^2$–dimensional invariant subspace and similarly labeled by total angular momentum values. (ii) go beyond the special orthogonal product reference $|G,G\rangle_z \otimes |H,H\rangle_x$ and study how well the encoding works for more general reference states (entangled, mixed or non-coherent spin states), thereby quantifying how imperfections in the quantum reference frame degrade the recovered coherences of $\mathcal{S}$.

### Appendix A: Background independent spin given two directional references

#### 1. Basis Change

To write the state of the system entirely in the $z$-basis we use the basis change (16), given by

$$|H,H\rangle_x^{\mathcal{H}} = \frac{1}{2^H}\sum_{h=-H}^{H}\sqrt{\binom{2H}{H+h}}\,|H,h\rangle_z^{\mathcal{H}}\,.$$

Replacing this in the state (14), which is given by

$$|\psi\rangle^{\mathcal{SGH}} = \Big(\alpha\,|G+\tfrac{1}{2},G+\tfrac{1}{2}\rangle + \frac{\beta}{\sqrt{2G+1}}\,|G+\tfrac{1}{2},G-\tfrac{1}{2}\rangle + \frac{\beta\sqrt{2G}}{\sqrt{2G+1}}\,|G-\tfrac{1}{2},G-\tfrac{1}{2}\rangle\Big)\otimes\frac{1}{2^H}\sum_{h=-H}^{H}\sqrt{\binom{2H}{H+h}}\,|H,h\rangle_z^{\mathcal{H}}$$

we obtain

$$|\psi\rangle^{\mathcal{SGH}} = \frac{1}{2^H}\sum_{h=-H}^{H}\sqrt{\binom{2H}{H+h}}\Big(\alpha\,|G+\tfrac{1}{2},G+\tfrac{1}{2}\rangle + \frac{\beta}{\sqrt{2G+1}}\,|G+\tfrac{1}{2},G-\tfrac{1}{2}\rangle + \frac{\beta\sqrt{2G}}{\sqrt{2G+1}}\,|G-\tfrac{1}{2},G-\tfrac{1}{2}\rangle\Big)\otimes|H,h\rangle_z^{\mathcal{H}}\,. \tag{A1}$$

Next, we do the basis change $\left\{|j^{\mathcal{SG}},m^{\mathcal{SG}}\rangle\right\}\otimes\left\{|j^{\mathcal{H}},m^{\mathcal{H}}\rangle\right\}\to\left\{|j^{\mathcal{SGH}},m^{\mathcal{SGH}},j^{\mathcal{SG}}\rangle\right\}$ with the Clebsch-Gordan coefficients defined in (17). This yields the following three terms.
First term:

$$\sum_{h=-H}^{H}\alpha\sqrt{\binom{2H}{H+h}}\,|G+\tfrac{1}{2},G+\tfrac{1}{2}\rangle\,|H,\,h\rangle = \sum_{h=-H}^{H}\alpha\sqrt{\binom{2H}{H+h}}\sum_{J=G+1/2+h}^{G+\frac{1}{2}+H}C^{J,G+1/2+h}_{G+1/2,G+1/2,h}\,|J,G+\tfrac{1}{2}+h,G+\tfrac{1}{2}\rangle\,. \tag{A2}$$

Second term:

$$\begin{aligned}&\sum_{h=-H}^{H}\frac{\beta}{\sqrt{2G+1}}\sqrt{\binom{2H}{H+h}}\,|G+\tfrac{1}{2},G-\tfrac{1}{2}\rangle\,|H,\,h\rangle\\ &=\sum_{h=-H}^{H}\frac{\beta}{\sqrt{2G+1}}\sqrt{\binom{2H}{H+h}}\sum_{J=G-1/2+h}^{G+1/2+H}C^{J,G-1/2+h}_{G+1/2,G-1/2,h}\,|J,G-\tfrac{1}{2}+h,G+\tfrac{1}{2}\rangle\,,\end{aligned} \tag{A3}$$

Third term:

$$\begin{aligned}&\sum_{h=-H}^{H}\frac{\beta\sqrt{2G}}{\sqrt{2G+1}}\sqrt{\binom{2H}{H+h}}\,|G-\tfrac{1}{2},G-\tfrac{1}{2}\rangle\,|H,\,h\rangle\\ &=\sum_{h=-H}^{H}\frac{\beta\sqrt{2G}}{\sqrt{2G+1}}\sqrt{\binom{2H}{H+h}}\sum_{J=G-1/2+h}^{G-1/2+H}C^{J,G-1/2+h}_{G-1/2,G-1/2,h}\,|J,G-\tfrac{1}{2}+h,G-\tfrac{1}{2}\rangle\,.\end{aligned} \tag{A4}$$

Their sum is the total state as given in (18):

$$\begin{aligned}|\psi\rangle^{\mathcal{SGH}} = \frac{1}{2^H}\sum_{h=-H}^{H}\sum_{J=G-1/2+h}^{G+1/2+H}\sqrt{\binom{2H}{H+h}}\Bigg[&\alpha\,C^{J,G+1/2+h}_{G+1/2,G+1/2,h}\,|J,G+\tfrac{1}{2}+h,G+\tfrac{1}{2}\rangle\\ &+\frac{\beta}{\sqrt{2G+1}}C^{J,G-1/2+h}_{G+1/2,G-1/2,h}\,|J,G-\tfrac{1}{2}+h,G+\tfrac{1}{2}\rangle\\ &+\frac{\sqrt{2G}\,\beta}{\sqrt{2G+1}}C^{J,G-1/2+h}_{G-1/2,G-1/2,h}\,|J,G-\tfrac{1}{2}+h,G-\tfrac{1}{2}\rangle\Bigg].\end{aligned} \tag{A5}$$

### 2. Group Averaging

Now, we will apply the incoherent group average to the state's density matrix $\rho^{\mathcal{SGH}}$ corresponding to the pure state $|\psi\rangle^{\mathcal{SGH}}$ given in (18). Explicitly, $\rho^{\mathcal{SGH}}$ is given by

$$\begin{aligned}\rho^{\mathcal{SGH}} = \frac{1}{2^H}\sum_{h=-H}^{H}\sum_{J=G-1/2+h}^{G+1/2+H}\sqrt{\binom{2H}{H+h}}\Bigg[&\alpha\, C^{J,G+1/2+h}_{G+1/2,G+1/2,h}\,|J,G+1/2+h,G+1/2\rangle \\ &+\frac{\beta}{\sqrt{2G+1}}C^{J,G-1/2+h}_{G+1/2,G-1/2,h}\,|J,G-1/2+h,G+1/2\rangle \\ &+\frac{\sqrt{2G}\,\beta}{\sqrt{2G+1}}C^{J,G-1/2+h}_{G-1/2,G-1/2,h}\,|J,G-1/2+h,G-1/2\rangle\Bigg] \\ \frac{1}{2^H}\sum_{h'=-H}^{H}\sum_{J'=G-1/2+h'}^{G+1/2+H}\sqrt{\binom{2H}{H+h'}}\Bigg[&\bar{\alpha}\,\bar{C}^{J',G+1/2+h'}_{G+1/2,G+1/2,h'}\,\langle J',G+1/2+h',G+1/2| \\ &+\frac{\bar{\beta}}{\sqrt{2G+1}}\bar{C}^{J',G-1/2+h'}_{G+1/2,G-1/2,h'}\,\langle J',G-1/2+h',G+1/2| \\ &+\frac{\sqrt{2G}\,\bar{\beta}}{\sqrt{2G+1}}\bar{C}^{J',G-1/2+h'}_{G-1/2,G-1/2,h'}\,\langle J',G-1/2+h',G-1/2|\Bigg].\end{aligned} \tag{A6}$$

Due to Schur's Lemma, when $J_1 \neq J_2$ and $m_1 \neq m_2$ the matrix elements $|J_1,m_1\rangle\langle J_2,m_2|$ vanish. The remaining, non-zero matrix elements, are those for which $J_1 = J_2$ and $m_1 = m_2$. When calculating $\mathcal{E}(\rho^{\mathcal{SGH}}) = \int_{SO(3)} \mathrm{d}\mu(\Omega)\ R^{\mathcal{SGH}}(\Omega)\ \rho^{\mathcal{SGH}}\ R^{\mathcal{SGH}}(\Omega)^\dagger$, this means that one has to match $J = J'$ for every element depending on the specific entry choose $h' = h$ or $h' = h \pm 1$. Performing the average then leads to (20)

$$\begin{aligned}\mathcal{E}(\rho^{\mathcal{SGH}}) = \sum_{J=G-1/2-H}^{G+1/2+H}\Bigg[&\frac{|\alpha|^2}{2}\,S^{(1)}_{JGH}|J,G+\tfrac{1}{2}\rangle\langle J,G+\tfrac{1}{2}| + \frac{\sqrt{2G}\alpha\bar{\beta}}{\sqrt{2G+1}}S^{(2)}_{JGH}|J,G+\tfrac{1}{2}\rangle\langle J,G-\tfrac{1}{2}| \\ &+\frac{G\,|\beta|^2}{2G+1}\,S^{(3)}_{JGH}|J,G-\tfrac{1}{2}\rangle\langle J,G-\tfrac{1}{2}| + \frac{\sqrt{2G}\,|\beta|^2}{2G+1}\,S^{(4)}_{JGH}|J,G+\tfrac{1}{2}\rangle\langle J,G-\tfrac{1}{2}| \\ &+\frac{\alpha\bar{\beta}}{\sqrt{2G+1}}S^{(5)}_{JGH}|J,G+\tfrac{1}{2}\rangle\langle J,G+\tfrac{1}{2}| + \frac{|\beta|^2}{2(2G+1)}\,S^{(6)}_{JGH}|J,G+\tfrac{1}{2}\rangle\langle J,G+\tfrac{1}{2}|\Bigg] + h.c.,\end{aligned}$$

with

$$\begin{aligned}S^{(1)}_{JGH} &= \frac{1}{4^H}\sum_{h=-H}^{H}\binom{2H}{H+h}|C^{J,G+1/2+h}_{G+1/2,G+1/2,h}|^2 \\ S^{(2)}_{JGH} &= \frac{1}{4^H}\sum_{h=-H}^{H}\sqrt{\binom{2H}{H+h}\binom{2H}{H+h+1}}C^{J,G+1/2+h}_{G+1/2,G+1/2,h}\bar{C}^{J,G+1/2+h}_{G-1/2,G-1/2,h+1} \\ S^{(3)}_{JGH} &= \frac{1}{4^H}\sum_{h=-H}^{H}\binom{2H}{H+h}|C^{J,G-1/2+h}_{G-1/2,G-1/2,h}|^2 \\ S^{(4)}_{JGH} &= \frac{1}{4^H}\sum_{h=-H}^{H}\binom{2H}{H+h}\bar{C}^{J,G-1/2+h}_{G-1/2,\,G-1/2,h}C^{J,G-1/2+h}_{G+1/2,G-1/2,h} \\ S^{(5)}_{JGH} &= \frac{1}{4^H}\sum_{h=-H}^{H}\sqrt{\binom{2H}{H+h}\binom{2H}{H+h+1}}C^{J,G+1/2+h}_{G+1/2,G+1/2,h}\bar{C}^{J,G+1/2+h}_{G+1/2,G-1/2,h+1} \\ S^{(6)}_{JGH} &= \frac{1}{4^H}\sum_{h=-H}^{H}\binom{2H}{H+h}|C^{J,G-1/2+h}_{G+1/2,G-1/2,h}|^2.\end{aligned} \tag{A7}$$

### 3. Large Gyroscope Limit

Note that even though the Clebsch-Gordan coefficient are all real numbers, we keep the complex conjugate notation for bookkeeping purposes. The Clebsch-Gordan coefficients are real numbers between -1 and 1 since they are inner products of normalized vectors. This implies that the sums $S^{(i)}_{JGH}$ defined above are also bounded by -1 from below and 1 from above:

$$\left|S^{(i)}_{JGH}\right| \leq \frac{1}{4^H}\sum_{h=-H}^{H}\binom{2H}{H+h} = \frac{1}{4^H}\cdot 2^{2H} = 1. \tag{A8}$$

Note that for $i = 2, 5$, the binomial coefficient is replaced with $\sqrt{\binom{2H}{H+h}\binom{2H}{H+h+1}}$, but this does not influence the limit, as can be confirmed numerically.

When $G = H \to \infty$, the matrix elements in $\mathcal{E}(\rho^{\mathcal{SGH}})$ are of the form $a^{(i)}_G S^{(i)}_{JGH}$. For $i = 4, 5, 6$, we have that $a^{(i)}_G \to 0$. Since $S^{(i)}_{JGH}$ are bounded, the corresponding matrix element vanishes, that is, $a^{(4,5,6)}_G S^{(4,5,6)}_{JGH} \to 0$.

For the remaining matrix elements $i = 1, 2, 3$, we denote $S^{(1,2,3)}_{JGH} \to S^{(1,2,3)}_{J\infty}$. We work under the assumption that this limit exists. We give numerical evidence that this is the case at the end of this Appendix. The incoherently group averaged state in the limit $G = H \to \infty$ is therefore given by

$$\begin{aligned}\mathcal{E}(\rho^{\mathcal{SGH}}) \to \mathcal{E}^{\lim}(\rho^{\mathcal{SGH}}) = \sum_{J=1/2}^{\infty}\Bigg[&\frac{|\alpha|^2}{2}\,S^{(1)}_{J\infty}|J, G+1/2\rangle\langle J;\, G+1/2| \\ &+ \frac{|\beta|^2}{2}\,S^{(2)}_{J\infty}|J, G-1/2\rangle\langle J, G-1/2| \\ &+ \alpha\bar{\beta}\,S^{(3)}_{J\infty}|J, G+1/2\rangle\langle J, G-1/2|\Bigg] + h.c.\end{aligned}$$

The letter $G$ remains here not as a variable, but only as part of the conventional label for the two different eigenstates of the operator $j^{\mathcal{SG}}$. Furthermore, looking at the expressions for $S^{(i)}_{JGH}$ one notices the numerically verifiable fact that for all $J$, $S^{(1)}_{J\infty} = S^{(2)}_{J\infty} = S^{(3)}_{J\infty}$ (see Fig. 3), which we denote as $S_{J\infty}$. We can therefore further compactify the expression for the limit physical state to the form (21),

$$\begin{aligned}\mathcal{E}^{\lim}(\rho^{\mathcal{SGH}}) &= \rho_{j^{\mathcal{SGH}}} \otimes \rho_{j^{\mathcal{SG}}} = \rho_{j^{\mathcal{SGH}}} \otimes |\psi_{j^{\mathcal{SG}}}\rangle\langle\psi_{j^{\mathcal{SG}}}|, \\ |\psi_{j^{\mathcal{SG}}}\rangle &= \alpha\,|G+1/2\rangle + \beta\,|G-1/2\rangle.\end{aligned} \tag{A9}$$

with the constant, numeric density matrix

$$\rho_{j^{\mathcal{SGH}}} = \sum_{J=1/2}^{\infty} S_{J\infty}|J\rangle\langle J|, \tag{A10}$$

describing the total angular momentum of the entire system $\mathcal{SGH}$. Its values $\langle J|\,\rho_{j^{\mathcal{SGH}}}\,|J\rangle = S_{J\infty}$ are depicted in Fig. 2 and show convergence to a delta function peaked at $J = \sqrt{2}\,G$ as $G \to \infty$ corresponding to the size of the sum of the two orthogonal spins $\mathcal{G}$ and $\mathcal{H}$. This constitutes numerical evidence for the existence of the limit $S_{J\infty}$.

## Appendix B: Coherent state characterisation of quantum reference frames

In this appendix we contrast the reference frame constructed out of gyroscopes used in the present work to the $L^2(\mathrm{SO}(3))$ reference frame used in the reference frame quantization procedure of [20]. We show that these are two different group coherent state systems [21], even though both allow for a full encoding of a qubit in the relative degrees of freedom between the reference frame and spin 1/2 particle. **TG:** The $L^2(G)$ reference frame is ideal, in the sense that it allows for a perfect encoding of the system of interest in the $G$-invariant algebra of the composite of reference and system. Examples of reference frame quantization in the literature which explore the relation between the size of the reference frame and the accuracy of the encoding, as done in the present work, include [22–24].

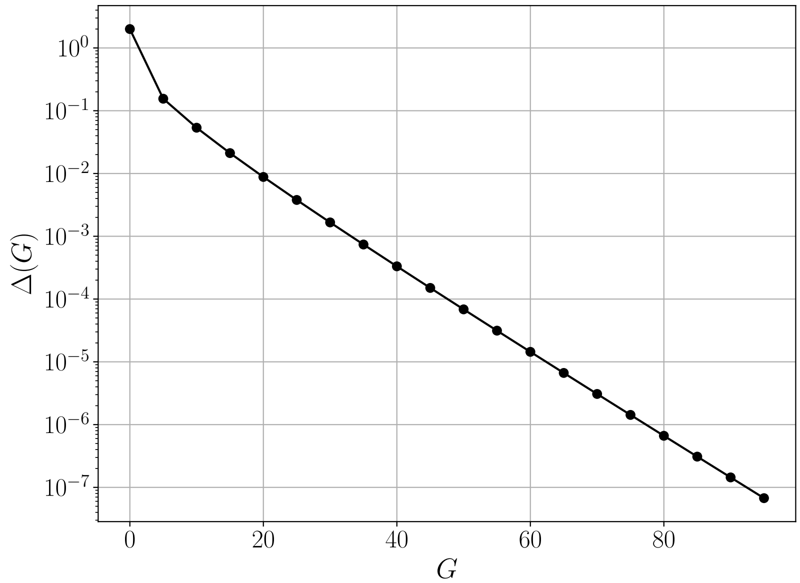


Figure 3. Numerical evaluation of $\Delta(G) = \sum_J (|S^{(1)}_{JGG} - S^{(2)}_{JGG}| + |S^{(1)}_{JGG} - S^{(3)}_{JGG}|)$ as a function of $G$. The decay of this value with increasing $G$ provides numerical evidence that $S^{(1)}_{JGG}$, $S^{(2)}_{JGG}$, and $S^{(3)}_{JGG}$ converge to a common asymptotic limit, denoted $S_{J\infty}$.

### 1. Ideal $L^2(\mathrm{SO}(3))$ coherent state systems

For a compact group $G$ the space of square integrable functions $L^2(G)$ (for a given choice of measure) transforms under the left regular representation as:

$$U_L(g)\,|g'\rangle = |gg'\rangle\,. \tag{B1}$$

Using the Peter-Weyl theorem we can decompose $L^2(G)$ under the left regular representation of $G$ as:

$$L^2(G) \simeq \bigoplus_{\lambda\in\hat{G}} V_\lambda \otimes W_\lambda, \tag{B2}$$

where $\hat{G}$ is the set of irreducible representations of $G$, $V_\lambda$ carries the irreducible representation $U_\lambda$ of $G$, and the *multiplicity* space $W_\lambda$ carries $\dim(W_\lambda)$ copies of the trivial representation. Moreover $\dim(V_\lambda) = \dim(W_\lambda)$. In the case of $G = \mathrm{SO}(3)$ the irreps are labeled by integers $\lambda$ and hence parametertising a rotation $\Omega \in \mathrm{SO}(3)$ by three real parameters $\alpha, \beta, \gamma$ gives:

$$U_L(\alpha,\beta,\gamma) \simeq \bigoplus_{j\in\mathbb{N}} D^j(\alpha,\beta,\gamma) \otimes \mathbb{1}_{2j+1}. \tag{B3}$$

The states $|g\rangle$ have support in every $j \in \mathbb{N}$ and hence the coherent state system transforms under $\bigoplus_{j\in\mathbb{N}} D^j(\alpha,\beta,\gamma) \otimes \mathbb{1}_{2j+1}$.

As shall be proven in the next section a key difference with the SO(3) coherent state system constructed in the present work is that the $L^2(\mathrm{SO}(3))$ coherent state system also carries a unitary representation of the right regular action, which in this case is the right regular representation:

$$U_R(g)\,|g'\rangle = |g'g^{-1}\rangle\,, \tag{B4}$$

where once more using the Peter-Weyl theorem:

$$U_R(\alpha,\beta,\gamma) \simeq \bigoplus_{j\in\mathbb{N}} \mathbb{1}_{2j+1} \otimes \bar{D}^j(\alpha,\beta,\gamma). \tag{B5}$$

### 2. Orthogonal gyroscope coherent state systems

#### a. Representation theoretic characterisation

The quantum reference frame of Sec. III is obtained from the reference state $|\psi(I)\rangle^{\mathcal{GH}} = |G,G\rangle_z^{\mathcal{G}} \otimes |H,H\rangle_x^{\mathcal{H}}$ acted on by the representation $D^G(\Omega)\otimes D^H(\Omega)$: $|\psi(\Omega)\rangle^{\mathcal{GH}} = D^G(\Omega)\otimes D^H(\Omega)\,|\psi(I)\rangle^{\mathcal{GH}}$, $\Omega\in \mathrm{SO}(3)$.

Using SU(2) representation theory we can decompose the representation as follows:

$$V^G \otimes V^H \simeq \bigoplus_{j=|G-H|}^{G+H} V^j, \tag{B6}$$
$$D^G(\Omega)\otimes D^H(\Omega) \simeq \bigoplus_{j=|G-H|}^{G+H} D^j(\Omega).$$

We now determine the support of the reference state $|G,G\rangle_z^{\mathcal{G}} \otimes |H,H\rangle_x^{\mathcal{H}}$ on the subspaces $V^j$ carrying representations $D^j$. Using standard angular momentum coupling rules we have:

$$\begin{aligned}|\psi(I)\rangle^{\mathcal{GH}} = |G,G\rangle_z^{\mathcal{G}} \otimes |H,H\rangle_x^{\mathcal{H}} &= |G,G\rangle_z^{\mathcal{G}} \otimes \left(\frac{1}{2^H}\sum_{h=-H}^{H}\sqrt{\binom{2H}{H+h}}\,|H,h\rangle_z^{\mathcal{H}}\right) \\ &\simeq \frac{1}{2^H}\sum_{h=-H}^{H}\sqrt{\binom{2H}{H+h}}\sum_{j=|G-H|}^{G+H} C^{j,G+h}_{G,G;H,h}\,|j,G+h\rangle \\ &= \frac{1}{2^H}\sum_{j=|G-H|}^{G+H}\sum_{h=-H}^{H}\sqrt{\binom{2H}{H+h}}C^{j,G+h}_{G,G;H,h}\,|j,G+h\rangle\,.\end{aligned} \tag{B7}$$

For every $j\in\{|G-H|,...,G+H\}$ one has a $h\in\{-H,...,H\}$ such that $j=G+h$, hence for every $j$ there is a coefficient $C^{j,G+h}_{G,G;H,h} = C^{j,j}_{G,G;H,h}\neq 0$. This shows that $|\psi(I)\rangle^{\mathcal{GH}}$ has support in every irrep $D^j$.

Since $|\psi(I)\rangle^{\mathcal{GH}}$ has support in every irreducible representation $D^j(R)$ for $j\in\{|G-H|,...,G+H\}$ the coherent state system $\{|\psi(\Omega)\rangle^{\mathcal{GH}}\,|\Omega\in\mathrm{SO}(3)\}$ transforms under the representation $D^G(\Omega)\otimes D^H(\Omega)\simeq\bigoplus_{j=|G-H|}^{G+H}D^j(\Omega)$ of SO(3).

To confirm that it is a SO(3) coherent state system we need to show that:

$$|\psi(\Omega)\rangle = e^{i\phi}\,|\psi(\Omega')\rangle \implies \Omega=\Omega', \forall \Omega,\Omega'\in\mathrm{SO}(3). \tag{B8}$$

Equivalently we need to show that the stabilizer group of the ray $|\psi(I)\rangle$ is trivial. It is sufficient to show that the stabilizer group of the vector $|\psi(I)\rangle$ is trivial.

The stabilizer of $|G,G\rangle_z^{\mathcal{G}}$ is the U(1) subgroup $\mathrm{Stab}(|G,G\rangle_z^{\mathcal{G}}) = D^G(e^{iZt}), t\in[0,2\pi)$ and the stabilizer of $|H,H\rangle_x^{\mathcal{H}}$ is the U(1) subgroup $\mathrm{Stab}(|H,H\rangle_x^{\mathcal{H}}) = D^G(e^{iXt}), t\in[0,2\pi)$. The stabilizer group $\mathrm{Stab}(|G,G\rangle_z^{\mathcal{G}}\otimes|H,H\rangle_x^{\mathcal{H}})$ of $|\psi(I)\rangle^{\mathcal{GH}} = |G,G\rangle_z^{\mathcal{G}}\otimes|H,H\rangle_x^{\mathcal{H}}$ , is given by $\mathrm{Stab}(|G,G\rangle_z^{\mathcal{G}})\cap\mathrm{Stab}(|H,H\rangle_x^{\mathcal{H}}) = D^G(I)\otimes D^H(I)$.

#### b. Left-right regular action

The SO(3) coherent state system $|\psi(\Omega)\rangle$ carries a left regular action (which is not the left regular representation however):

$$D^G(\Omega)\otimes D^H(\Omega)\,|\psi(\Omega')\rangle = |\psi(\Omega\Omega')\rangle\,, \forall\Omega,\Omega'\in\mathrm{SO}(3). \tag{B9}$$

We now show that the right regular action:

$$|\psi(\Omega')\rangle \mapsto |\psi(\Omega'\Omega^{-1})\rangle\,, \tag{B10}$$

is not unitary.

Let us assume there exists a unitary representation $C : \Omega \mapsto C(\Omega)$ such that $C(\Omega) |\psi(\Omega')\rangle = |\psi(\Omega'\Omega^{-1})\rangle$. We now show that this leads to a contradiction.

By construction this representation commutes with $D(\Omega) = D^G(\Omega) \otimes D^H(\Omega)$. By Schur's lemma this implies that:

$$C(\Omega) \simeq \bigoplus_{j=|G-H|}^{G+H} c_j(\Omega)\mathbb{1}^j, c_j(\Omega) \in \mathbb{C}\ \forall j \in \{|G-H, ..., G+H\}. \tag{B11}$$

By unitarity $c_j(\Omega) = e^{i\theta_j(\Omega)}$ for all $\forall j \in \{|G-H, ..., G+H\}, \Omega \in \mathrm{SO}(3)$.

Eq. (B10) implies that $C(\Omega) |\psi(I)\rangle = |\psi(\Omega^{-1}\rangle$. Define $\Pi_j$ the projector onto $V_j$, the carrier space of the irreducible representation $D_j$. Then $\Pi_j(C(\Omega) |\psi(I)\rangle) = c_j(\Omega)\Pi_j(|\psi(I)\rangle)$. However by assumption $D(\Omega^{-1}) |\psi(I)\rangle = C(\Omega) |\psi(I)\rangle$ which implies $\Pi_j(D(\Omega^{-1}) |\psi(I)\rangle) = D^j(\Omega^{-1})\Pi_j |\psi(I)\rangle = c_j(\Omega)\Pi_j(|\psi(I)\rangle)$. This implies the one dimensional subspace spanned by $\Pi_j(|\psi(I)\rangle)$ is invariant under $D^j(\Omega)$ which contradicts the fact that $D^j$ is irreducible and of dimension strictly greater than 1 for $j \neq 0$.

### 3. Reference frame quantization for generic coherent state system

We now consider a reference frame $\mathcal{H}_R$ consisting of some coherent state system for $G$: $\{|\psi(g)\rangle\, |g \in G\}$ which is not ideal: $|\langle\psi(g)|\psi(g')\rangle|^2 \neq 0$ for $g \neq g'$. We moreover assume that:

$$\int |\psi(g)\rangle\langle\psi(g)|\ \mathrm{d}g = N\mathbb{1}, \quad N \in \mathbb{R}. \tag{B12}$$

The encoding maps are:

$$\begin{aligned} \rho_S \mapsto \tilde{\rho}_{\mathrm{inv}} :=& \frac{1}{N d_R} \int_G U_S(g)\rho_S U_S^\dagger(g) \otimes |\psi(g)\rangle\langle\psi(g)|_R\ \mathrm{d}g, \\ E_S \mapsto \tilde{E}_{\mathrm{inv}} :=& \frac{1}{N} \int_G U_S(g) E_S U_S^\dagger(g) \otimes |\psi(g)\rangle\langle\psi(g)|_R\ \mathrm{d}g, \end{aligned} \tag{B13}$$

where $d_R = \dim(\mathcal{H}_R)$.

The probabilities for the encoded states and effects are:

$$\begin{aligned} \mathrm{Tr}(\rho_{\mathrm{inv}} E_{\mathrm{inv}}) =& \frac{1}{d_R N^2} \mathrm{Tr}\left[\int_G U_S(g)\rho_S U_S^\dagger(g) \otimes |\psi(g)\rangle\langle\psi(g)|_R\ \mathrm{d}g \quad \int_G U_S(h) E_S U_S^\dagger(h) \otimes |\psi(h)\rangle\langle\psi(h)|_R\ \mathrm{d}h)\right] \\ =& \frac{1}{d_R N^2} \int_G \int_G \mathrm{Tr}\left[U_S(g)\rho_S U_S(g^{-1}h) E_S U_S^\dagger(h) \otimes |\psi(g)\rangle \langle\psi(g)|\psi(h)\rangle \langle\psi(h)|_R\right]\ \mathrm{d}g\ \mathrm{d}h, \end{aligned} \tag{B14}$$

we carry out the change of integration variable: $g^{-1}h \to k$ and $g \to g$:

$$= \frac{1}{d_R N^2} \int_G \int_G \mathrm{Tr}\left[U_S(g)\rho_S U_S(k) E_S U_S^\dagger(gk) \otimes |\psi(g)\rangle \langle\psi(g)|\psi(gk)\rangle \langle\psi(gk)|_R\right]\ \mathrm{d}g\ \mathrm{d}k, \tag{B15}$$

since $|\psi(h)\rangle_R$ form an overcomplete basis we use $\mathrm{Tr}_R(\rho_{RS}) = \frac{1}{N}\int \langle\psi(h)|_R\, \rho_{RS}\, |\psi(h)\rangle_R\ \mathrm{d}h$:

$$\begin{aligned} &= \frac{1}{d_R N^2} \int_G \int_G \mathrm{Tr}\left[U_S(g)\rho_S U_S(k) E_S U_S^\dagger(gk)\right] \times \frac{1}{N} \int_G \langle\psi(h)|\ |\psi(g)\rangle \langle\psi(g)|\psi(gk)\rangle \langle\psi(gk)|_R\ |\psi(h)\rangle\ \mathrm{d}h\ \mathrm{d}g\ \mathrm{d}k \\ &= \frac{1}{d_R N^2} \int_G \int_G \mathrm{Tr}\left[U_S(g)\rho_S U_S(k) E_S U_S^\dagger(gk)\right] \times \langle\psi(g)|\psi(gk)\rangle \langle\psi(gk)|\ \frac{1}{N}\left[\int_G |\psi(h)\rangle \langle\psi(h)|\ \mathrm{d}h\right]\ |\psi(g)\rangle\ \mathrm{d}g\ \mathrm{d}k \\ &= \frac{1}{d_R N^2} \int_G \int_G \mathrm{Tr}\left[U_S(g)\rho_S U_S(k) E_S U_S^\dagger(gk)\right] \langle\psi(g)|\psi(gk)\rangle \langle\psi(gk)|\psi(g)\rangle\ \mathrm{d}g\ \mathrm{d}k, \\ &= \frac{1}{d_R N^2} \int_G \int_G |\langle\psi(e)|\psi(k)\rangle|^2\ \mathrm{Tr}\left[U_S(g)\rho_S U_S(k) E_S U_S^\dagger(gk)\right]\ \mathrm{d}g\ \mathrm{d}k, \end{aligned} \tag{B16}$$

where $e \in G$ is the identity element. Using cyclicity of trace, $\mathrm{Tr}\left[U_S(g)\rho_S U_S(k) E_S U_S^\dagger(gk)\right] = \mathrm{Tr}\left[\rho_S U_S(k) E_S U_S^\dagger(k)\right]$. Moreover, by tracing (B12) we get that $\int_G \mathrm{d}g = d_R N$, leading to

$$\begin{aligned} &= \frac{1}{N} \int_G |\langle\psi(e)|\psi(k)\rangle|^2\ \mathrm{Tr}\left[\rho_S U_S(k) E_S U_S^\dagger(k)\right]\ \mathrm{d}k \\ &= \mathrm{Tr}\left(\rho_S \int_G \mathrm{d}k\ \frac{|\langle\psi(e)|\psi(k)\rangle|^2}{N} U_S(k) E_S U_S^\dagger(k)\right). \end{aligned} \tag{B17}$$

This is a trace of $\rho_S$ multiplying the effect $E_S$ averaged over all the coherent states of the system with their weights $\frac{|\langle\psi(e)|\psi(k)\rangle|^2}{N}$, $\int_G \mathrm{d}k\, \frac{|\langle\psi(e)|\psi(k)\rangle|^2}{N} = 1$ being their overlap with the coherent state corresponding to the identity element of the group.

We note that in the case where the reference frame is ideal: $|\langle\psi(g)|\psi(g')\rangle|^2 = 0$ for $g \neq g'$ and $N = 1$, we obtain an exact encoding: $\mathrm{Tr}(\rho_S E_S) = \mathrm{Tr}(\rho_{\mathrm{inv}} E_{\mathrm{inv}})$.

**TG:**

$$g \neq g' \implies |\psi(g)\rangle \neq |\psi(g')\rangle \tag{B18}$$


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