Quantum Permutations and 'Genuinely Quantum' Reference Frames

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Quantum permutations, or magic unitaries, have in recent years been explored in the context of identifying 'genuinely quantum' isometries of graphs. Here, we import this tool in physics, showing that quantum permutations yield a generalisation of quantum reference frames in a discrete setting that is reminiscent of the passage from special to general relativity. We show that the typical quantum reference frames framework corresponds to quantum permutations classified as 'classical' in the mathematical literature, and which we demonstrate are quantum controlled transformations (superpositions of classical coordinate maps). Genuinely quantum permutations (i) allow to construct *non–commuting* quantum reference frames (ii) correspond to *local*, as opposed to global, superpositions of transformations. Strikingly, we find that the non-commutativity of quantum fields, when used as reference systems, is exactly what implies that the change of frame is achieved through a genuine quantum permutation. We illustrate the above with several examples in both first and second quantization formalism, which demonstrate (a) simultaneous control on non–commuting variables, (b) the existence of bipartite states that can be localized with a genuine quantum permutations and cannot be localized with the usual quantum reference frame transformations, (c) extension of the Ising model symmetries to genuinely quantum permutations, and (d) extension of the symmetries of a scalar field action on curved spacetime to genuinely quantum permutations. While we have in mind applications in quantum gravity, we expect our formalism to be of interest in a wide range of topics in quantum information.

I. INTRODUCTION

Coordinates have played a crucial role in the development of modern physics. A choice of coordinates is made in order to describe a physical system. We may interpret this choice as a relational description of a physical system of interest with respect to another physical system—which realizes the coordinate system— a system of reference. If we demand that all systems obey quantum mechanics, we are led to a description of a quantum system with respect to another quantum system.

This is the logic developed in the program of quantum reference frames, with preliminary ideas appearing since the 80s [1–3]. More recently, inspired by the field of quantum correlations with indefinite causal ordering e.g. [4–6], significant development has taken place using quantum information tools e.g. [7–15].

While the applicability of a framework of quantum reference frames is potentially very wide, its development is especially oriented toward applications in quantum gravity. The system which serves as the reference, can be taken to be spacetime itself. In a quantum theory of gravity, spacetime will obey quantum mechanics. Then, it should come with a relational understanding of how to describe quantum matter with respect to a quantum spacetime, as there isn't anything else 'behind spacetime' to use as reference. Similarly to how Einstein posited that physical laws should be invariant under changes of coordinate systems, it is natural to expect that in a quantum theory of spacetime background independence will be implemented as the invariance under changes of some kind of quantum coordinates.

Coordinates can be tied to the idea of a reference system, for instance, by taking the map that defines the coordinates to correspond to something of physical substance that takes different values at different spacetime locations—a test field. Considering this to be a quantum field, leads to a notion of quantum coordinates [16], an idea explored also in the context of attempts at formulating a quantum equivalence principle [17, 18].

However, the notions of quantum reference frames and quantum coordinates which have appeared in the literature seem to correspond to a small subset of the possibilities: in essence, they correspond to quantum superpositions of classical transformations, formed by quantum controlling on a reference system. They are, in a sense, 'not very quantum'. How to go beyond, while maintaining that the transformations are unitary, has not been obvious.

In this work, we show that the mathematical theory of quantum permutations is remarkable in that it shows how this can be achieved concretely, and that this shortcoming has been noticed completely independently of physics. Tellingly, we will show that quantum permutations that are not of the quantum-controlled form have been termed 'genuinely quantum' by mathematicians (hence the title of this work), reflecting the general intuition that truly quantum transformations should inherently involve non-commutativity—here, of the orthogonal projections that enter the definition of quantum permutations.

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Quantum permutations, or magic unitaries, are studied in operator theory as a natural generalization of permutations to the quantum realm, see [19] for a review. The use of permutations implies that we will be working in a discrete setting. Here, we will focus on finite dimensional Hilbert spaces (although many of our results automatically extend to the countable case). Quantum permutations extend the classical symmetries of graphs to quantum symmetries: some pairs of graphs are classically not isomorphic, but are mapped to each other by a quantum permutation. In this sense, these new symmetries are quantum symmetries, as they have no classical analogue. In the context of non-local games on graphs, it has been shown that sometimes a classical winning strategy does not exist but only a quantum one, precisely when the graphs are not classically isomorphic but are quantum isomorphic [20]. Here, we import these striking results in physics using the language of quantum information. We will see that *quantum permutations arise naturally in quantum theory as transformations between relational descriptions, and allow for local quantum reference frame choices that are point-wise non-commuting.*

The material is organized as follows. We show that quantum permutations are the transformations that allow to switch between descriptions of quantum systems with respect to other quantum systems. In standard quantum mechanics the quantum permutations act on the elements of the orthonormal basis defined by some relational observable (Section II A). In the case of discrete quantum fields, quantum permutations act on locations (Section II B). Crucially, the non-commutativity of reference quantum fields implies that the change of frame is achieved through a genuine quantum permutation.

The above motivate importing the technology of quantum permutations to physics. For this, we translate them in the language of quantum information, where they are naturally interpreted as acting on a state space composed of two factors, the systems of interest and the reference systems (Section III). We show that the quantum permutations known as 'classical' in the mathematical literature correspond to quantum controlled transformations (Section IV). This demonstrates in particular that the quantum permutations known as 'genuinely quantum' in the mathematical literature are those that cannot be cast in a quantum controlled form. Hereafter, we introduce the physics oriented terminology quantum controlled (QC) permutations for what are known as the 'classical' quantum permutations and beyond quantum controlled (BQC) permutations for those called 'genuine' quantum permutations. We explore within our formalism the properties of building blocks of BQC permutations in Section V.

Strikingly, BQC permutations correspond to *local* superpositions of transformations, while QC permutations are *global* superpositions of transformations (Section VI). Any BQC permutation can be understood as making an independent choice of basis for quantum control at each point; taken together, these choices yield a transformation that is not QC, since arbitrary local basis choices will generically fail to commute. This suggests that the passage from QC to BQC transformations allows for a generalisation of quantum reference frames akin to the passage from global coordinate changes to local coordinate changes, reminiscent of the passage from special to general relativity.

To build intuition on BQC permutations, we examine an example of a genuinely quantum graph symmetry using a simple physical model of a spin on a graph (Section VII). This motivates understanding quantum permutations as symmetries of the dynamics, even when a graph structure is not present. We show that usual quantum reference frame transformations studied in the literature, which are symmetries of Hamiltonians of certain kinds, can be cast as QC permutations (Section VIII). We extend the quantum reference frames framework using BQC permutations by giving examples of (a) transformations and Hamiltonians that are not quantum controlled (Section IX A), (b) a bipartite state that can be localized with a BQC permutation and cannot be localized with a QC permutation (Section IX B), and (c) compare with related relevant literature in Section IX C. In Section X, we see that the above imply the existence of new, genuinely quantum, symmetries of the Ising model. Finally, in Section XI we show that quantum permutations that are differentiable in a discrete sense, are new, genuinely quantum, symmetries of the discretised action of a scalar field on a curved spacetime.

II. QUANTUM PERMUTATIONS ARE NECESSARY FOR RELATIONAL DESCRIPTIONS AMONG QUANTUM SYSTEMS

We begin by a demonstration that quantum permutations (introduced formally in the following Section) emerge naturally as reference system transformations when considering quantum theory in a *discrete and relational* setting. We first discuss (A) first-quantization systems and then (B) quantum fields. We will see that the role of QPs as reference system transformations or a sort of quantum coordinate maps becomes especially clear when considering quantum fields.

A. Quantum permutations from quantum mechanics

Consider the standard first-quantization picture of a system S described by a Hilbert space $\mathcal{H}_S$ and an observable (Hermitian operator) $\hat{X}_1$ on $\mathcal{H}_S$. In a relational approach, the eigenvalues of the observable $\hat{X}_1$ are not absolute, but, given in relation to some reference system O_1. In turn, assuming the reference to be a quantum system as well, it comes with its own Hilbert space $\mathcal{H}_{O_1}$. The standard first-quantization picture can then be understood as the system and reference described by a joint state space $\mathcal{H}_S \otimes \mathcal{H}_{O_1}$, with the observable $\hat{X}_1$ extended trivially (as the identity) on the sector $\mathcal{H}_{O_1}$. That is, $\hat{X}_1$ is understood as $\hat{X}_1 \otimes \mathbb{1}$ acting on $\mathcal{H}_S \otimes \mathcal{H}_{O_1}$.

Now, we ask: *what is the description of the physical property of S corresponding to $\hat{X}_1$, if the reference system is not O_1 but some other quantum system O_2 with a state space $\mathcal{H}_{O_2}$*? This will correspond to some Hermitian $\hat{X}_2$, acting on $\mathcal{H}_S \otimes \mathcal{H}_{O_2}$, and will not act trivially on $\mathcal{H}_{O_2}$. This is because it will depend on *both* the (eigen)values of $\hat{X}_1$ *and* the state of O_2.

We now formally define the above and show that the general form of $\hat{X}_2$ is a conjugation of $\hat{X}_1$ by a quantum permutation. To avoid confusion, we stress that $\hat{X}_1$ and $\hat{X}_2$ refer to the *same* relational physical quantity, for instance a relative position, as seen from the perspective of two different reference frames or observers, O_1 and O_2.

Let S be a system of interest with $\mathcal{H}_S$ its N-dimensional Hilbert space, and $\hat{X}_1$ an observable on $\mathcal{H}_S$ with eigenvalues $x_1 = x_1^{(1)}, ..., x_1^{(N)}$, given with respect to a reference system $\mathcal{H}_{O_1}$. Let O_2 be an additional reference system and $\mathcal{H}_{O_2}$ Hilbert space. The total Hilbert space is thus $\mathcal{H}_S \otimes \mathcal{H}_{O_1} \otimes \mathcal{H}_{O_2}$. *A relative observable with respect to the reference* O_2, describing the same physical quantity as $\hat{X}_1$, is an observable $\hat{X}_2$ on $\mathcal{H}_S \otimes \mathcal{H}_{O_2}$ with eigenvalues $x_2 = x_2^{(1)}, ..., x_2^{(N)}$ such that (1) For every x_1 there exists an observable $\hat{X}_{2,x_1}$ such that $\hat{X}_2 |x_1\rangle |\phi\rangle = |x_1\rangle \otimes \hat{X}_{2,x_1} |\phi\rangle$ for all $|\phi\rangle \in \mathcal{H}_{O_2}$, and (2) For all $x_1 \neq x_1'$, $|\phi\rangle, |\phi'\rangle \in \mathcal{H}_{O_2}$ and x_2, if $\hat{X}_2 |x_1\rangle |\phi\rangle = x_2 |x_1\rangle |\phi\rangle$ and $\hat{X}_2 |x_1'\rangle |\phi'\rangle = x_2 |x_1'\rangle |\phi'\rangle$, then $\langle\phi|\phi'\rangle = 0$.

These two simple conditions ensure that X_2 is a 'good' relational observable. Condition 1 ensures that on eigenstates of $\hat{X}_1$, the corresponding (relative to O_2) observable $\hat{X}_2$ only acts non-trivially on the state of O_2. Intuitively, that when the state of S from the point of view of O_1 is definite, whether it becomes indefinite after the transformation depends only on the state of O_2.

Condition 2 ensures that eigenvalues relative to O_2 distinguish eigenvalues relative to O_1. In particular, it states that if two different eigenvalues of $\hat{X}_1$ are transformed to the same eigenvalue of $\hat{X}_2$, this is only due to the states of O_2 being distinguishable. Intuitively, this condition ensures that no information about the system is lost when changing frames (it is analogous to the bijectivity assumption of the following sub-Section).

Now, it can be shown that any observable (Hermitian) $\hat{X}_2$ satisfies conditions 1 and 2 (it is a relative observable with respect to O_2) if and only if

$$\hat{X}_2 = u \; X_1 \; u^\dagger \tag{1}$$

for some

$$u = \sum_{x_1, x_2} |x_1\rangle \langle x_2|_S \otimes u_{x_1 x_2,\, O_2}. \tag{2}$$

satisfying[1]

$$\begin{aligned} u_{x_1 x_2} &= u^\dagger_{x_1 x_2} = u^2_{x_1 x_2} \\ \sum_{x_2} u_{x_1 x_2} &= \sum_{x_1} u_{x_1 x_2} = \mathbb{1}_{O_2}. \end{aligned} \tag{3}$$

The proof is given in Appendix A. Condition 1 gives the first set of equations and the decomposition of identity with respect to x_2. Condition 2 yields the decomposition of identity with respect to x_1. As we discuss in the next Section, (3) is the definition of a *quantum permutation* u.

To give an example, consider S to be a particle on a line with $\hat{X}_1$ its position relative to some reference observer (e.g. an experimenter or apparatus) O_1, and O_2 another observer on the same line. The relative observable $\hat{X}_2$ distinguishes the locations of S as measured by O_2. Concretely, take $\hat{X}_2$ to be the signed distance between S and O_2, that is, $\hat{X}_2 = \hat{X}_1 - \hat{X}_{O_2}$, where $\hat{X}_{O_2}$ is the position of O_2 relative to O_1. Given two states with the same $\hat{X}_2$ eigenvalue but different $\hat{X}_1$ eigenvalues we have $\langle\phi|\phi'\rangle = 0$ because $|\phi\rangle$ and $|\phi'\rangle$ are eigenstates of $\hat{X}_{O_2}$ with different eigenvalues (satisfying Condition 2). Anticipating what follows, note this simple example concerns a special case of quantum permutations, a quantum controlled (QC) transformation. It is extended to an example of a beyond quantum controlled (BQC) transformation in Section IX A.

We have seen that the defining properties of the quantum mechanical transformation between the relative to O_1 observable $\hat{X}_1$ and the relative to O_2 observable $\hat{X}_2$, is a quantum permutation. This follows from simple conditions that assure a well defined relational description between quantum systems.

B. Quantum permutations from quantum fields

We now consider that both the system of interest and the reference systems with respect to which it is described, are an abstract sort of discrete 'quantum fields'. Here, by quantum field we simply mean a family of observables $\hat{\varphi}(q)$ where q are elements of some abstract finite set $\mathcal{M}$. No further structure is assumed in the formal setting of this Section. In later Sections, we take q to be the nodes of a graph or lattice, q play the role of a discrete analogue of the points of a spacetime manifold. The reader is forewarned that below whenever we write $\hat{\varphi}_1$ and $\hat{\varphi}_2$, we do not mean that there are two fields, but that this is the same abstract field $\hat{\varphi}$ expressed in different (quantum) coordinates. Similarly, when we write x_1

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[1] Note a subtlety: $\hat{X}_2$ is invariant to compositions of u with arbitrary unitaries on the reference sector $\mathcal{H}_{O_1} \otimes \mathcal{H}_{O_2}$ (which act trivially on S). Nevertheless, the essence of the transformation is captured in the quantum permutation u, which acts non-trivially on the system of interest S.

and x_2, we refer to different coordinates of some abstract point q. Also, note that in the previous sub-Section we did not have an analogue of the abstract field $\hat{\varphi}$ (which is defined on abstract points q), as in quantum mechanics there is no analogue of the abstract set $\mathcal{M}$.

Consider three quantum fields $\hat{\varphi}(q)$, $\hat{\chi}_1(q)$ and $\hat{\chi}_2(q)$. These describe, respectively, the system of interest $\hat{\varphi}$ and two reference systems O_1, O_2. We define the *relative field* $\hat{\varphi}_i(x^\mu)$ as describing $\hat{\varphi}$ relative to $\hat{\chi}_i$, where x^μ is an eigenvalue of $\hat{\chi}_i$. Namely, the operator $\hat{\varphi}_i(x^\mu)$ will coincide with the operator $\hat{\varphi}(q)$ on states where the reference field $\hat{\chi}_i(q)$ takes the value x^μ.

More precisely, consider a quantum field $\hat{\varphi}(q)$ that takes arbitrary states in some (Fock) space $\mathcal{H}_\varphi$. This is the system of interest. Consider also two reference quantum fields $\mathcal{H}_{O_i}$ for $i = 1, 2$, each composed of 4 quantum scalar fields $\hat{\chi}_i(q) = \left(\hat{\chi}_i^{(\mu)}(q)\right)_{\mu=0,1,2,3}$ for $q \in \mathcal{M}$. The spectra of the operators $\hat{\chi}_i(q)$ are thus points $x^\mu \in \mathbb{R}^4$. The total Hilbert space is $\mathcal{H} = \mathcal{H}_\varphi \otimes \mathcal{H}_{O_1} \otimes \mathcal{H}_{O_2}$. By a relative to O_i quantum field $\hat{\varphi}_i$, we mean that there is a (discrete) subset of $\mathbb{R}^4$ such that for all $|\psi\rangle \in \mathcal{H}$ with $\hat{\chi}_i(q)\,|\psi\rangle = x^\mu\,|\psi\rangle$, it holds that $\hat{\varphi}_i(x^\mu)\,|\psi\rangle = \hat{\varphi}(q)\,|\psi\rangle$.

For $\hat{\varphi}_i$ to be well defined, O_i need to be good reference systems. For this, we restrict to subsets of $\mathcal{M}$ and $\mathcal{H}_{O_i}$ such that $\mathcal{H}_{O_i}$ only contain states of the reference fields that have enough variability to uniquely name all points in $\mathcal{M}$. This means states for which the reference field is 'bijective' in the following sense: there exists a family of operators $\hat{\chi}_i^{-1}(x^\mu)$ on the spectrum of all $\hat{\chi}_i(q)$ such that $\hat{\chi}_i(q)\,|\phi_i\rangle = x^\mu\,|\phi_i\rangle$ if and only if $\hat{\chi}_i^{-1}(x^\mu)\,|\phi_i\rangle = q\,|\phi_i\rangle$.[2] In a sense, this is a simple 'quantum version' of coordinate maps. Note that relative fields $\hat{\varphi}_i(x^\mu)$ would be analogous to the usual fields of quantum field theory if $\hat{\chi}(q)$ were classical coordinate maps.

Now, define P_i^{qx} to be the orthogonal projection on the x^μ eigenspace of $\hat{\chi}_i(q)$. It holds that $\sum_{x^\mu} P_i^{qx} = \mathbb{1}_{O_i}$, $P_i^{qx} = P_i^{qx,\dagger}$ and $P_i^{qx} P_i^{qx'} = \delta_{xx'} P_i^{qx}$. Crucially, due to the 'bijectivity' assumption that O_i are good reference quantum fields, P_i^{qx} is then also the orthogonal projection on the q eigenspace of $\hat{\chi}_i^{-1}(x^\mu)$. This means that it *also* holds that $\sum_q P_i^{qx} = \mathbb{1}_{O_i}$. That is, a summation on any of the upper indices of the orthogonal projections P_i^{qx} yields the identity. This is the characteristic property of quantum permutations.

Let us find the field $\hat{\varphi}_i(x^\mu)$ explicitly. For all $q \in \mathcal{M}$ and x^μ we have,

$$\hat{\varphi}_i(x^\mu) P_i^{qx} = \hat{\varphi}(q) \otimes P_i^{qx}. \tag{4}$$

Summing over q, we get $\hat{\varphi}_i(x^\mu) \sum_q P_i^{qx} = \sum_q \hat{\varphi}(q) \otimes P_i^{qx}$. From the bijectivity assumption it follows that $\sum_q P_i^{qx} = \mathbb{1}_{O_i}$, which yields the relative field

$$\hat{\varphi}_i(x^\mu) = \sum_q \hat{\varphi}(q) \otimes P_i^{qx}. \tag{5}$$

Then, we have

$$\hat{\varphi}_2(x_2^\mu) = \sum_{x_1^\mu} \hat{\varphi}_1(x_1^\mu)\, u_{x_2 x_1} = \sum_{x_1^\mu} u_{x_2 x_1}\, \hat{\varphi}_1(x_1^\mu) \tag{6}$$

with

$$u_{x_2 x_1} = \mathbb{1}_S \otimes \sum_q P_1^{qx_1} \otimes P_2^{qx_2}. \tag{7}$$

As in Section II A, the $u_{x_2 x_1}$ satisfy

$$\begin{aligned} u_{x_2 x_1} = u_{x_2 x_1}^\dagger = u_{x_2 x_1}^2 \\ \sum_{x_1} u_{x_2 x_1} = \sum_{x_2} u_{x_2 x_1} = \mathbb{1}_{O_1 O_2}, \end{aligned} \tag{8}$$

which are the defining properties of a quantum permutation (see next Section). Therefore, with minimal assumptions on the reference systems to enable well-defined relative descriptions of otherwise arbitrary systems of interest, it follows that the transformation between relative fields is given by a quantum permutation.

To make contact with the formalism of the previous subsection where we discussed first-quantisation, note that another formal way to write the transformation is as the matrix multiplication:

$$\begin{pmatrix} \hat{\varphi}_2(x_2^{(1)}) \\ \hat{\varphi}_2(x_2^{(2)}) \\ \vdots \\ \hat{\varphi}_2(x_2^{(N)}) \end{pmatrix} = \begin{pmatrix} u_{x_2^{(1)} x_1^{(1)}} & \cdots & u_{x_2^{(1)} x_1^{(N)}} \\ u_{x_2^{(2)} x_1^{(1)}} & \cdots & u_{x_2^{(2)} x_1^{(N)}} \\ \vdots & \ddots & \vdots \\ u_{x_2^{(N)} x_1^{(1)}} & \cdots & u_{x_2^{(N)} x_1^{(N)}} \end{pmatrix} \begin{pmatrix} \hat{\varphi}_1(x_1^{(1)}) \\ \hat{\varphi}_1(x_1^{(2)}) \\ \vdots \\ \hat{\varphi}_1(x_1^{(N)}) \end{pmatrix}, \tag{9}$$

where recall that $x_i^{(k)}$ are the eigenvalues of the reference field $\hat{\chi}_i(q)$.

Now, let us make two important remarks. First, it is not only the transformation u that fulfills the definition of a quantum permutation (QP), but also the sets of orthogonal projections P_i. While the transformation u transforms between the fields relative to each reference — $\hat{\varphi}_1$ and $\hat{\varphi}_2$ — the operators P_i^{qx} encode the relation between the abstract field $\hat{\varphi}$ and each of the relative fields $\hat{\varphi}_i$. In this sense, the P_i can be thought of as a quantum analogue to coordinate maps, and the quantum permutation u as a quantum coordinate change (it encodes how to switch between coordinates assigned by O_1 and those assigned by O_2).[3]

[2] Since q is generally not a complex number, we abuse the notation $q\,|\phi_i\rangle$ to mean $\lambda_q\,|\phi_i\rangle$, with λ_q some fixed injection of $\mathcal{M}$ into $\mathbb{C}$.

[3] Interestingly, the relation (7) that produces a QP u from two QPs P_1, P_2 is a known operation in the mathematical literature called the Woronowicz product [19, 21].

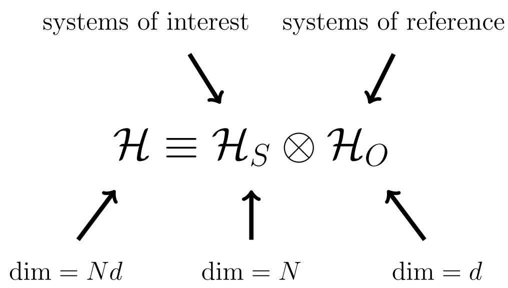

Figure 1. The names and dimensions of the Hilbert space on which magic unitaries act. We regard $\mathcal{H}_X$ as a system of interest. It contains the eigenbasis $|x\rangle$ of the observable X. This basis is the one permuted. $\mathcal{H}_O$ can be regarded as an external observer, as is the case in QRFs.

Second, if the reference fields do not commute at two points q and q', the corresponding orthogonal projectors will also not commute. That is,

$$[\hat{\chi}_i(q), \hat{\chi}_i(q')] = 0 \Leftrightarrow [P_i^{qx_i}, P_i^{q'x'_i}] = 0 \quad \forall x_i, x'_i \tag{10}$$

Since it is a characteristic property of quantum fields that they may not commute at different points (also known as the principle of micro-causality), this implies that generic quantum reference frame transformations will involve such non–commuting projectors. This implies that the quantum permutation that does changes from the frame of one field to the other must be 'genuinely quantum'.

III. QUANTUM PERMUTATIONS

Having seen that quantum permutations arise naturally when considering relational descriptions among quantum systems, we now give their precise definition (see [19] for further detail) and express them in the bra-ket notation.

Take $\mathcal{H}_O$ a separable Hilbert space, whose dimension we denote d, which could be either finite or infinite. A *quantum permutation*, or *magic unitary* u is a collection $(u_{xy})_{1\leq x,y\leq N}$ of bounded operators $u_{xy} \in \mathcal{B}(\mathcal{H}_O)$, with N finite, which are orthogonal projections that partition $\mathcal{H}_O$ into orthogonal subspaces for a constant x or y, arranged as follows. The operators u_{xy} compose the quantum permutation u by understanding the indexes x, y as signifying rows and columns arranged in a 'matrix of matrices' u with entries u_{xy}. Note that while u is a unitary, the 'entries' u_{xy} are orthogonal projections and generally not unitaries.

That the operators u_{xy} are orthogonal projections is equivalent to the condition

$$u_{xy}^\dagger = u_{xy} = u_{xy}^2. \tag{11}$$

That the u_{xy} partition the entire $\mathcal{H}_O$ in orthogonal subspaces is equivalent to demanding

$$\sum_z u_{xz} = \sum_z u_{zy} = \mathbb{1}_O \quad \forall x, y. \tag{12}$$

Then, since each u_{xy} is to be understood as an entry in a matrix of matrices u, the above condition dictates that each row and each column sums up to the identity operator of $\mathcal{H}_O$.

It follows from the above definition that for $a \neq b$ and for all x, y, the pairs u_{xa} and u_{xb}, as well as the pairs u_{ay} and u_{by}, are disjoint projections. That is,

$$\begin{aligned} u_{xa}u_{xb} &= 0 \\ u_{ay}u_{by} &= 0. \end{aligned} \tag{13}$$

This is due to a more general property of projections that we will be using often: projections $p_i^2 = p_i$ that sum up to the identity, $\sum_i p_i = \mathbb{1}$ are necessarily mutually orthogonal, $p_i p_j = 0$ for $i \neq j$. In particular, this implies that all entries of the same row or the same column of a quantum permutation commute. An example of a quantum permutation with four rows and columns is

$$u = \begin{pmatrix} \pi_1 & \mathbb{0}_O & \mathbb{1}_O - \pi_1 & \mathbb{0}_O \\ \mathbb{1}_O - \pi_1 & \mathbb{0}_O & \pi_1 & \mathbb{0}_O \\ \mathbb{0}_O & \pi_2 & \mathbb{0}_O & \mathbb{1}_O - \pi_2 \\ \mathbb{0}_O & \mathbb{1}_O - \pi_2 & \mathbb{0}_O & \pi_2 \end{pmatrix} \tag{14}$$

with π_i orthogonal projections on $\mathcal{H}_O$.

Quantum permutations are a generalization of a permutation matrix in the following sense. A permutation matrix has all zeros and a single unit in any row and column. So, all rows and columns of a permutation matrix sum up to one. A quantum permutation retains and generalizes this feature: all rows and columns sum to the unit *matrix*. For this reason, quantum permutations are also known as 'magic unitaries' (since ordinary permutation matrices are 'magic squares'). When the entries u_{xy} are one by one matrices, the definition of a quantum permutation coincides with ordinary permutations. When all entries u_{xy} commute among them, the quantum permutation is called 'classical' and when at least one pair u_{xy} and $u_{x'y'}$ does not commute it is called 'genuine', for reasons that will become clear in the following Sections.

To understand the action of quantum permutations on quantum mechanical states, it is useful to express them in the bra-ket notation. By inspection of the definition of quantum permutations, they are operators on a Hilbert space of the form

$$\mathcal{H} \equiv \mathcal{H}_S \otimes \mathcal{H}_O. \tag{15}$$

where $\mathcal{H}_S$ is N–dimensional, $\mathcal{H}_O$ is d–dimensional, and $\mathcal{H}$ is therefore of dimension Nd. Later on, the factor $\mathcal{H}_S$ will be the space of the systems of interest and the factor $\mathcal{H}_O$ the space of the reference systems or 'observers', see Figure 1. In the next Section, where we consider the special case of quantum controlled transformations, $\mathcal{H}_S$ is the target space and $\mathcal{H}_O$ the control space.

To give an example of the mechanics of the formalism, consider any state $|\psi\rangle \in \mathcal{H}$. This can be written as

$$|\psi\rangle = \sum_{x=1}^{N} \psi(x) |x\rangle |\phi_x\rangle \in \mathcal{H} \tag{16}$$

with $|x\rangle \in \mathcal{H}_S$ an eigenvector of some Hermitian $\hat{X} \otimes \mathbb{1}$, $\psi(x)$ a (discrete) normalised wavefunction, and $|\phi_x\rangle$ some arbitrary normalized vector in $\mathcal{H}_O$. In the terminology of Section II A, $\hat{X}$ is a relative observable corresponding to the observer with respect to which the tensor product structure was defined (as it acts trivially on $\mathcal{H}_O$), and $|\phi_x\rangle$ is some joint state of the reference systems. Applying u results in

$$u |\psi\rangle = \sum_x |x\rangle \otimes \sum_{x'} \psi(x') \, u_{xx'} \, |\phi'_x\rangle \,. \tag{17}$$

In matrix form, this reads

$$= \begin{pmatrix} u_{11} & \dots & u_{1N} \\ \vdots & & \vdots \\ u_{N1} & \dots & u_{NN} \end{pmatrix} \begin{pmatrix} \psi(1) |\phi_1\rangle \\ \vdots \\ \psi(N) |\phi_N\rangle \end{pmatrix}. \tag{18}$$

A physics-oriented notation for writing a quantum permutation that we found convenient in calculations is (2), repeated here for the reader's convenience

$$u = \sum_{x,x'} |x\rangle \langle x'| \otimes u_{xx'}.$$

Having reviewed their definition and basic properties, in the following two Sections we investigate how quantum permutations act on quantum mechanical states.

IV. QUANTUM CONTROLLED PERMUTATIONS

The simplest extension of the concept of permutations to the quantum realm is to consider quantum controlled permutations. These are superpositions of classical permutations. Assume σ_k are K classical permutations matrices on N items.[4] That is, σ_k is an $N \times N$ matrix with a single unit entry in each row and column and remaining entries are zero. Given π_k orthogonal projections $\pi_k^\dagger = \pi_k = \pi_k^2$ on $\mathcal{H}_O$ which sum up to the identity $\sum_k \pi_k = \mathbb{1}_O$, we define a *quantum controlled (QC) permutation* as a unitary of the form

$$u = \sum_{k=1}^{K} \sigma_k \otimes \pi_k. \tag{19}$$

[4] Note that K is a different unrelated number than N and d. The dimension of the total space is still taken to be Nd.

The $\sigma_k \otimes \pi_k$ are matrices of matrices that look like the permutation matrix σ_k, but with the operator π_k inserted in the place of the ones and the zero operator inserted in the place of the zeros. When restricted to any subspace $\pi_k \mathcal{H}_O$ of $\mathcal{H}_O$, the permutation is definite — it is σ_k. In this context, we can think of $\mathcal{H}_S$ from Section II A as the target space and $\mathcal{H}_O = \mathcal{H}_{O_1} \otimes \mathcal{H}_{O_2}$ as the control space.

The subset of quantum controlled permutations can be characterized in a different way. It is immediate that all entries in a quantum permutation of the form (19) commute, as all π_k commute. The opposite direction is not so straightforward but can be shown, we give the proof in Appendix D. That is, *QC permutations are exactly all quantum permutations whose entries all commute.* This proves that QC permutations are the subset of QPs known as 'classical QPs' in the mathematical literature.

Let us see this with a simple example. Consider a quantum permutation u where the entries u_{xy} are two by two matrices. In contrast to ordinary permutations that only have one non zero entry per row and column, it is now possible to have *two* non zero entries: each will project on a one dimensional orthogonal subspace, and the sum of both is the two by two identity matrix. It is this behavior that allows one to encode the intuitive requirement, for example, of 'permuting 1 with 2 in quantum superposition of permuting 1 with 3'. This superposition of classical permutations is the quantum permutation

$$\begin{aligned} u &= \begin{pmatrix} \mathbb{0}_O & |\uparrow\rangle\langle\uparrow| & |\downarrow\rangle\langle\downarrow| \\ |\uparrow\rangle\langle\uparrow| & |\downarrow\rangle\langle\downarrow| & \mathbb{0}_O \\ |\downarrow\rangle\langle\downarrow| & \mathbb{0}_O & |\uparrow\rangle\langle\uparrow| \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \otimes |\uparrow\rangle\langle\uparrow| + \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \otimes |\downarrow\rangle\langle\downarrow| , \end{aligned} \tag{20}$$

where $\mathcal{H}_O = \mathbb{C}^2$ is spanned by the orthonormal basis $|\uparrow\rangle, |\downarrow\rangle$.

The reason quantum permutations whose entries u_{xy} commute is called a 'classical' permutation in the context of graph symmetries [19] can be seen from (19): every graph that has such a quantum controlled permutation as a symmetry will also have each of the ordinary permutations σ_k as symmetries as well (it will have more than one classical symmetries, and the QC symmetry simply corresponds to a superposition of those). In the mathematical literature, it is only when any of the entries of the quantum permutation matrix *do not commute* that we say that we have a '*genuinely quantum permutation*'. For example, in (14) it corresponds to the requirement that $[\pi_1, \pi_2] \neq 0$. For an explicit example of a genuinely quantum symmetry of a graph see Section VII.

We have been presented with a nomenclature clash in mathematics and physics. Transformations of the quantum controlled form (19) are generally thought to be 'quantum' in physics. We have shown that the quantum permutations defined by entries (orthogonal projectors) which are all commuting are exactly those of the form (19). These have been called 'classical permutations' in the mathematical theory of quantum permutations, because the 'hallmark of quantumness' in that context is considered to be the non–commutativity of the orthogonal projectors composing the quantum permutation matrix. Instead of 'classical permutations', we have termed them here quantum controlled (QC) permutations. Correspondingly, when at least two entries of the quantum permutation do not commute, instead of 'genuinely quantum permutations' we introduce the terminology *beyond quantum controlled (BQC) permutations.*

V. BEYOND QUANTUM CONTROLLED PERMUTATIONS

We examined so far the special case of quantum controlled permutations. These were a simple extension of classical transformations via the principle of quantum superposition. We now turn to understanding the general case of quantum permutations.

Consider the following BQC permutation for $N = 4$ and d arbitrary[5]

$$u = \begin{pmatrix} \pi_1 & \mathbb{1}_O - \pi_1 & \mathbb{0}_O & \mathbb{0}_O \\ \mathbb{1}_O - \pi_1 & \pi_1 & \mathbb{0}_O & \mathbb{0}_O \\ \mathbb{0}_O & \mathbb{0}_O & \pi_2 & \mathbb{1}_O - \pi_2 \\ \mathbb{0}_O & \mathbb{0}_O & \mathbb{1}_O - \pi_2 & \pi_2 \end{pmatrix}, \tag{21}$$

with $\pi_1\pi_2 \neq \pi_2\pi_1$. Recall that in order for this to be a quantum permutation it also holds that $\pi_k = \pi_k^2 = \pi_k^\dagger$. BQC permutation of the form (21) are building blocks of more general BQC permutations, much like any permutation group can be written in terms of 2-cycles [22].

As discussed in the previous Section and proved in Appendix D, any quantum permutation with at least two non-commuting entries *cannot* be put in the form (19). Therefore, this quantum permutation cannot be understood as a controlled application of classical permutations on the $N = 4$ elements it acts upon.

Now, we can rewrite u as

$$\begin{aligned} u = &\begin{pmatrix} \mathbb{1}_2 & \mathbb{0}_2 \\ \mathbb{0}_2 & \mathbb{0}_2 \end{pmatrix} \otimes \pi_1 + \begin{pmatrix} \sigma & \mathbb{0}_2 \\ \mathbb{0}_2 & \mathbb{0}_2 \end{pmatrix} \otimes (\mathbb{1}_O - \pi_1) \\ &+ \begin{pmatrix} \mathbb{0}_2 & \mathbb{0}_2 \\ \mathbb{0}_2 & \mathbb{1}_2 \end{pmatrix} \otimes \pi_2 + \begin{pmatrix} \mathbb{0}_2 & \mathbb{0}_2 \\ \mathbb{0}_2 & \sigma \end{pmatrix} \otimes (\mathbb{1}_O - \pi_2), \end{aligned} \tag{22}$$

where $\sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is the 2×2 non-trivial permutation. We see that u is a sum of four terms; in each term, the left factor is acting on the 4–dimensional $\mathcal{H}_S$ and the right factor is acting on the d–dimensional $\mathcal{H}_O$. Written in this way, it resembles a situation where we think of $\mathcal{H}_O$ as the control space.

Recall that in the usual quantum information notion of a quantum controlled transformation, we would have a sum over terms where the left factors are unitaries on a target space $\mathcal{H}_S$. The right factors would be orthogonal projections that sum up to the identity on the control space $\mathcal{H}_O$.

However, there are two important differences when considering a BQC permutation. First, in (22) the operators acting on the target space $\mathcal{H}_S$ are *not* unitaries on $\mathcal{H}_S$ — they are unitaries on *subspaces* of $\mathcal{H}_S$. Second, the control space $\mathcal{H}_O$ is not acted upon with complementary projections. The pair π_1 and $\mathbb{1}_O - \pi_1$ sum to the identity, as does the pair π_2 and $\mathbb{1}_O - \pi_2$. However, *all* four of them appear in (22), while, for instance, π_1 and π_2 do *not* commute.

Intuitively, (22) can be thought of as *two* quantum controlled transformations packaged together in one unitary, which is not overall a quantum controlled transformation. The first two terms correspond to a (unitary) quantum controlled operation on the subspace given by restricting to the 'upper' half of $\mathcal{H}_S$ and the last two terms correspond to a (unitary) quantum controlled operation on the subspace given by restricting to the 'lower' half of $\mathcal{H}_S$.

It is this insight that yields the remarkable property that two unitary operators sum up to a unitary: in general, the sum of two unitaries is not a unitary. In the next Section, we see how this can be used to construct arbitrary BQC transformations. We will see in the next Section that in a certain sense this allows to construct local superpositions of transformations (while the quantum controlled case corresponds to global superpositions of transformations).

Let us now calculate explicitly how a BQC permutation acts on a basis of $\mathcal{H}$ using a simple toy-model. Take $N = 4$ and $d = 2$. Imagine a particle described by a wavefunction giving an amplitude to be in any of four positions, and $\mathcal{H}_O$ to be the 2-dimensional state space of some reference system spanned by $|\uparrow\rangle, |\downarrow\rangle$. Consider the quantum permutation (21) with $\pi_1 = |\uparrow\rangle\langle\uparrow|$ and $\pi_2 = |+\rangle\langle+|$, with $|\pm\rangle := \frac{1}{\sqrt{2}}\Big(|\uparrow\rangle \pm |\downarrow\rangle\Big)$. This completely defines the quantum permutation. To understand its action, it is sufficient to examine how it acts on the eight elements of a basis of $\mathcal{H}$:

$$\begin{aligned} u\,|1\rangle\,|\uparrow\rangle &= |1\rangle\,|\uparrow\rangle \\ u\,|2\rangle\,|\uparrow\rangle &= |2\rangle\,|\uparrow\rangle \\ u\,|1\rangle\,|\downarrow\rangle &= |2\rangle\,|\downarrow\rangle \\ u\,|2\rangle\,|\downarrow\rangle &= |1\rangle\,|\downarrow\rangle \\[1em] u\,|3\rangle\,|+\rangle &= |3\rangle\,|+\rangle \\ u\,|4\rangle\,|+\rangle &= |4\rangle\,|+\rangle \\ u\,|3\rangle\,|-\rangle &= |4\rangle\,|-\rangle \\ u\,|4\rangle\,|-\rangle &= |3\rangle\,|-\rangle\,. \end{aligned} \tag{23}$$

[5] A technical counting argument yields that BQC permutations only exist for $N \geq 4$ [19].

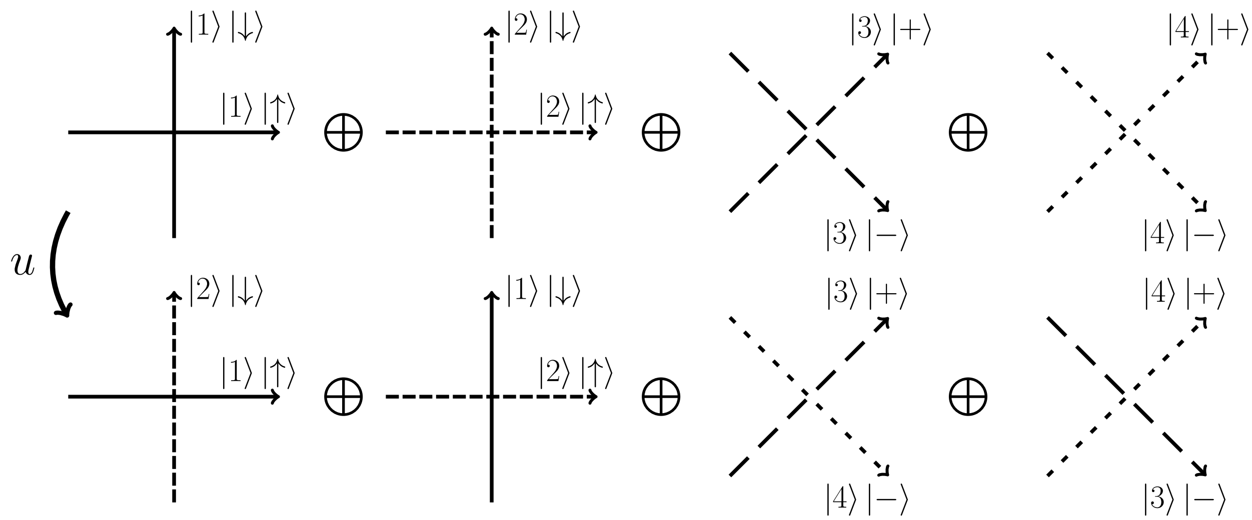

Figure 2. An illustration of the action of the BQC permutation u from (21) with $\pi_1 = |\uparrow\rangle\langle\uparrow|, \pi_2 = |+\rangle\langle+|$ on the Hilbert space $\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_O = \bigoplus_{x=1}^4 |x\rangle \otimes \mathcal{H}_O$, with $\mathcal{H}_O = \mathbb{C}^2$. That is, $N = 4$, $d = 2$ and the total Hilbert space is of dimension $Nd = 8$. The figure illustrates how the different subspaces of each original copy of $\mathcal{H}_O$ move from one component to the other, creating a different decomposition of $\mathcal{H}$ into orthogonal subspaces.

The left hand side above is u acting on a basis element of $\mathcal{H}$ and the right hand side shows the resulting state. We see that u exchanges the labels 1 and 2 when O is in $|\uparrow\rangle$, and does nothing to them when it is in $|\downarrow\rangle$. A similar thing happens for 3 and 4, but in the $\pm$ basis instead of the $\uparrow\downarrow$ basis. We see here explicitly that the transformation of the first four basis elements corresponds to a quantum controlled operation on the subspace given by restricting to the 'upper' half of $\mathcal{H}_S$ spanned by $|1\rangle$ and $|2\rangle$, and the action on the last four basis elements corresponds to a quantum controlled operation on the subspace given by restricting to the 'lower' half of $\mathcal{H}_S$ spanned by $|3\rangle$ and $|4\rangle$. This example of a BQC permutation is depicted in Fig. 2.

With these tools in hand, we now turn to the construction of arbitrary quantum permutations.

VI. LOCAL RELATIVITY OF SUPERPOSITION

In the previous Section we remarked that the matrix structure and properties of BQC permutations enables to appropriately sum quantum controlled unitaries, packaging them into larger unitaries. We will now build on this insight to show that *BQC (QC) transformations are formed by local (global) superpositions of transformations.*

First, note that the quantum controlled of (21) is block diagonal. This can be generalised to a recipe to construct quantum permutations, by the observation that transformations of the following form are unitaries:

$$u = \sum_{b=1}^{B} \sum_{k=1}^{K_b} \sigma_k^{(b)} \otimes \pi_k^{(b)}. \tag{24}$$

Here, $\pi_k^{(b)}$ are orthogonal projections and for each b, $\sum_{k=1}^{K_b} \pi_k^{(b)} = \mathbb{1}_O$. For each b, the matrices $\sigma_k^{(b)}$ each contain a permutation on a block specific to b and zeroes elsewhere. Each two such b-blocks are disjoint and together form a partition of $1, ..., N$. The quantum permutation (22) is an example with $B = 2, K_1 = K_2 = 2$.[6] This property will be used in Section IX A to construct beyond quantum controlled quantum reference frames.

Second, note that when $B = N$, each 'block' is d-dimensional, the dimension of the observers or reference sector $\mathcal{H}_O$. In this case, we can write any QP in the following form

$$u = \sum_{x}^{N} \sum_{k=1}^{K_x} |y_k(x)\rangle\langle x| \otimes \pi_k^{(x)}. \tag{25}$$

This means that *any QP can be understood as a quantum controlled transformation at every point.* Indeed, every point x is transformed to a new point $y_k(x)$, controlled by the orthogonal projections $\pi_k^{(x)}$. In the case where these all commute, the permutation is QC. That is, if all the $\pi_k^{(x)}$ commute, a common eigenbasis exists for every k and x and (25) reduces to (19). Namely, a set of π'_k exists so that

$$\begin{aligned} u_{QC} &= \sum_{x}^{N} \sum_{k=1}^{K} |y_k(x)\rangle\langle x| \otimes \pi'_k \\ &= \sum_{k}^{K} \sigma_k \otimes \pi'_k \end{aligned} \tag{26}$$

[6] This way to construct 'genuinely quantum permutations' corresponds to the *disjoint isomorphism criterion* found in [23].

where here $\sigma_k = \sum_x^N |y_k(x)\rangle\langle x|$

In this sense, a BQC permutation is a *local* superposition of transformations (controlling over a different basis of $\mathcal{H}_O$ at every point) and a QC permutation is a *global* superposition of transformations (controlling over one basis of $\mathcal{H}_O$ everywhere). We see concretely how this takes place in the first-quantisation formalism when the locations are graph nodes in Section VII, and more generally when the locations are positions on a line in Section IX A. As remarked in Section II B, and as we will see in more detail in Sections IX B and XI, the fact that BQC permutations are locally defined superpositions of transformations arises from the non-commutativity of reference fields.

VII. QUANTUM PERMUTATIONS AS SYMMETRIES

We now see a first example of how 'genuinely quantum' symmetries of graphs can yield new physical symmetries. For this, we use a toy model of a particle on a graph, which demonstrates intuitively the properties of BQC permutaitons discussed in Sections V and VI.

A prominent role of QPs in mathematics is that they introduce the notion of quantum symmetries of graphs, also known as quantum automorphisms. The definition of a quantum automorphism of a graph Γ is a QP u such that $u_{xy}u_{x'y'} = 0$ if $x \sim_\Gamma x'$ and $y \nsim_\Gamma y'$ (or vice versa). Here, $\sim_\Gamma$ signifies whether two nodes of the graph Γ are connected by an edge. That is, recalling that u_{xy} and $u_{x'y'}$ are orthogonal projections, u preserves the graph structure if the subspace in which x goes to y is orthogonal to the subspace in which x' goes to y'. This is equivalent to demanding that the adjacency matrix of the graph A_Γ is invariant under u. That is, u is a quantum automorphism of Γ if and only if $A_\Gamma \otimes \mathbb{1}_O = u\left(A_\Gamma \otimes \mathbb{1}_O\right) u^\dagger$.

A classical automorphism (classical symmetry) of Γ is a permutation σ_k for which $A_\Gamma = \sigma_k A_\Gamma \sigma_k^\dagger$. If Γ has more than one classical automorphisms, using (19) it is immediate to construct a QC permutation u that is a quantum automorphism of Γ, namely $u = \sum_{k=1}^K \sigma_k \otimes \pi_k$, for any set of orthogonal projections π_k that sum up to the identity. This is the reason that QC permutations are called 'classical' permutations in the mathematical literature: they correspond to a combination of classical symmetries of graphs; they are not 'new' symmetries. Quantum automorphisms that are BQC permutations are therefore a non-trivial extension of symmetries of graphs: strikingly, pairs of graphs can have a BQC permutation as its symmetry while it has *none* classical symmetry. Therefore, *BQC permutations allow to construct symmetries with no classical counterpart.*

Imagine a quantum particle $\mathcal{H}_S$ and two reference systems $\mathcal{H}_{O_1}, \mathcal{H}_{O_2} \simeq \mathbb{C}^2$. The adjacency matrix of the graph can be thought of as an observable (Hermitian operator) given relative to O_1:

$$A = \sum_{x \sim_\Gamma x'} |x\rangle\langle x'| \otimes \mathbb{1}_{O_1 O_2} = A_\Gamma \otimes \mathbb{1}_{O_1 O_2}. \tag{27}$$

Here, Γ is a labeled graph with the node labels x corresponding to some fixed basis $|x\rangle$ of $\mathcal{H}_S$, and we understand them here as positions relative to O_1 (similar to Section II A).

The general state of the particle can be written as $|\psi\rangle = \sum_x \psi(x)|x\rangle \otimes |\phi_x\rangle$ where $\psi(x)$ is the (discrete) wavefunction and $|\phi_x\rangle \in \mathcal{H}_{O_1} \otimes \mathcal{H}_{O_2}$ are states of the reference systems. The physical meaning of A can be seen from its expectation value. Because the adjacency matrix is symmetric, for every edge between the nodes i, j we get a term $\langle\phi_i|\phi_j\rangle + \langle\phi_j|\phi_i\rangle$, and so

$$\langle\psi|A|\psi\rangle = \frac{1}{2} \sum_{i \sim_\Gamma j} \mathrm{Re}\langle\phi_i|\phi_j\rangle. \tag{28}$$

Then, the observable A gives a measure of the alignment of the spin state across the edges on the graph. If the state and graph are such that each spin state pair connected by an edge is aligned (anti–aligned) on the two nodes then $\langle\psi|O|\psi\rangle$ is maximised (minimised), and it is zero if each pair of spins connected by an edge is orthogonal.

Given a state $|\psi\rangle$, all states generated by acting on $|\psi\rangle$ with isomorphisms of the graph will give the same expectation value for the A. These are the classical symmetries of the graph Γ, classical permutations σ on $\mathcal{H}_S$ that preserve the edge structure of Γ. For these, $\sigma^\dagger A_\Gamma \sigma = A_\Gamma$, and thus $(\sigma \otimes \mathbb{1}_{O_1O_2})^\dagger A\,(\sigma \otimes \mathbb{1}_{O_1O_2}) = A$. Quantum controlled magic unitaries $u = \sum_{k=1}^K \sigma_k \otimes \pi_k$ with σ_k being symmetries of Γ also leave A invariant, so that these magic unitaries extend its symmetry group.

BQC permutations that are symmetries of Γ non-trivially extend the set of transformations leaving A invariant. An example is the BQC permutation u studied in Section V (see (23), (21) and Figure 2):

$$u = \begin{pmatrix} |\uparrow\rangle\langle\uparrow|_{O_2} & |\downarrow\rangle\langle\downarrow|_{O_2} & & \\ |\downarrow\rangle\langle\downarrow|_{O_2} & |\uparrow\rangle\langle\uparrow|_{O_2} & & \\ & & |+\rangle\langle+|_{O_2} & |-\rangle\langle-|_{O_2} \\ & & |-\rangle\langle-|_{O_2} & |+\rangle\langle+|_{O_2} \end{pmatrix}. \tag{29}$$

This is a BQC symmetry, for instance, of the 4-cycle (square) graph.

As discussed in Section V, this BQC permutation exchanges nodes 1 and 2 (3 and 4) on the down (plus) subspace and leaves them unchanged on the up (minus) subspace. Intuitively, the quantum permutation (29) quantum controls 'simultaneously' on two non-commuting bases of the reference system $\mathcal{H}_{O_2}$ on different 'parts' of the graph. This demonstrates the point discussed in Section VI: the characteristic difference between the special case of QC and general case of BQC permutations is that the latter are constructed through non–commuting bases of the subspaces defined by different locations. In

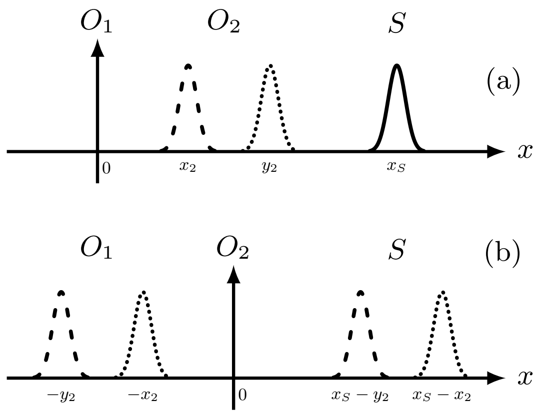

Figure 3. A simple use case for quantum controlled reference frame transformations. (a) A state from O_1's perspective: O_1 is at the origin, S is at a fixed position, and O_2 is in a superposition of two locations. (b) Same situation described from O_2's perspective after applying the quantum reference frame transformation: O_2 is at the origin, while O_1 and S are in an entangled state such that signed distances are preserved. Dashed and dotted lines represent different superposition branches.

the above example, on the nodes $1, 2$ the control basis is $|\uparrow, \downarrow\rangle$ and on the nodes $3, 4$ it is the $|\pm\rangle$.

The toy-model studied in this Section used a first-quantisation formalism. In Section X, we will use the second-quantisation formalism to see how genuinely quantum symmetries of graphs are new symmetries for the Ising model. In the following two Sections, we turn to demonstrating that quantum permutations generate new symmetries also in physically relevant contexts where a graph structure is not present.

VIII. QUANTUM CONTROLLED PERMUTATIONS AS QUANTUM REFERENCE FRAMES

In this section we consider transformations developed in the context of the program of *quantum reference frames (QRFs)*, e.g. [8–11, 16, 24–29], which is a main inspiration for this work. As the name suggests, this type of transformations is used to move between descriptions relative to different quantum systems. We will then see that the transformations that have been typically considered in the literature can be cast in the form of quantum controlled permutations. In the next Section, we see how to leverage the theory of quantum permutations to extend the quantum reference frame apparatus to beyond quantum controlled transformations.

A QRF transformation is usually defined given a symmetry group G and transforms the state of a tripartite system $\mathcal{H}_S \otimes \mathcal{H}_{O_1} \otimes \mathcal{H}_{O_2}$ from the frame of O_1 to that of O_2. The systems O_1 and O_2 are the observer/reference systems and S is the system of interest. To briefly introduce QRFs, let us take the simple example when the group G is the translations. Imagine a situation as depicted in Figure 3a. O_1 is in a definite location at $x = 0$. From its perspective, S is in a definite position $x = x_S$ and O_2 is in a superposition of locations x_2 and y_2. The state is

$$|\psi\rangle^{O_1} = |x_S\rangle_S \otimes |0\rangle_{O_1} \otimes \frac{1}{\sqrt{2}}\Big(|x_2\rangle + |y_2\rangle \Big)_{O_2}. \qquad (30)$$

Quantum reference frame transformations allow to 'jump' to the perspective of O_2, even though O_2 is in a quantum state (in a superposition of locations). That is, a QRF transformation changes the description so that the location of O_2 becomes definite (say at $x = 0$) while the (signed) distances, a relational observable, is conserved. This encodes the presence of translational symmetry. The only state that preserves the distances while O_2 is at $x = 0$ is

$$|\psi\rangle^{O_2} = \frac{1}{\sqrt{2}}\Big(|x_S - x_2\rangle_S \, |{-x_2}\rangle_{O_1} + |x_S - y_2\rangle_S \, |{-y_2}\rangle_{O_1} \Big) \qquad (31)$$
$$\otimes |0\rangle_{O_2}$$

which is depicted in Fig. 3b. By inspection, we see that the transformation which sends $|\psi\rangle^{O_2} \to |\psi\rangle^{O_1}$ will control on each of the superposed locations of O_2, and perform the required translation to the other systems accordingly, in such a way as to preserve the distances.

This is called a QRF transformation, and can be written in general as [27]

$$S^G_{O_1 \to O_2} = \mathrm{SWAP}_{O_1 O_2} \sum_{g \in G} T^\dagger_{g,S} \otimes |g^{-1}\rangle \langle g|_{O_2}\,, \qquad (32)$$

where $|g\rangle := T_g |e\rangle$ for all $g \in G$ form an orthonormal basis constructed by acting with G on a fixed state $|e\rangle$ with the left regular representation T_g.[7] In the example discussed above, taking G to be the translation group the QRF transformation (32) sends the state (30) to the state (31). It is important to notice that every T_g is a *permutation* of the basis elements $|g\rangle$: $T_g |g'\rangle = |gg'\rangle$, that is, T_g takes basis elements to basis elements and is invertible. The operator $\mathrm{SWAP}_{O_1 O_2}$ exchanges the O_1 and O_2 elements of the tensor product.[8]

The QRF transformation (32) can be cast in the form of a QC permutation (up to a unitary on the observers' sector), since it can be written as

$$S^G_{O_1 \to O_2} = \Big(\mathbb{1} \otimes \tilde{V} \Big) \sum_{g \in G} T^\dagger_g \otimes |g\rangle \langle g|_{O_2}\,. \qquad (33)$$

[7] We leave implicit that T_g are formally different operators on each Hilbert space, i.e. we omit writing $T^{O_1}_g$, $T^{O_2}_g$ and T^S_g.

[8] Note that while it is common to define QRF transformations by fixing a symmetry group G, this is not necessary for the usual formal definition. It is sufficient to fix a basis $|g\rangle_S$ for $\mathcal{H}_S$, and bases $|g\rangle_{O_1,O_2}$ to control over. The T_g can be taken to be unitaries on $\mathcal{H}_S$ that permute the basis elements $|g\rangle_S$. The additional flexibility here is because quantum permutations are formed by choosing a *different basis* at each point, see (25).

Here, $\tilde{V} = \text{SWAP}_{O_1O_2} \sum_g |g^{-1}\rangle \langle g|_{O_2}$ is a unitary only on the observers' sector O_1 and O_2.

Given a Hamiltonian, any quantum permutation that is a superposition of permutations that are symmetries of the Hamiltonian is itself a symmetry. For example, in a $P+2$ partite system $S_1, S_2...S_P, O_1, O_2$ any central force Hamiltonian of the form

$$H = \sum_i \frac{\hat{p}^2_{S_i}}{2m_{S_i}} + \sum_{i\neq j} V\left(\left|\hat{x}_{S_i} - \hat{x}_{S_j}\right|\right) \tag{34}$$

with m_{S_i} the mass of particle S_i retains its form under translational QRF transformations [27]. It is therefore invariant under the corresponding QC permutation of the form (33).

We have seen that QC permutations, understood as QRF transformations, can be symmetries of the dynamics of a physical system. We now proceed to give examples of 'genuinely quantum' reference frame transformations, using BQC permutations.

IX. 'GENUINELY QUANTUM' REFERENCE FRAMES

In the previous Section we saw that typical QRF transformations studied in the literature can be cast in the form of quantum controlled permutations. However, as discussed toward the end of Section II B, the non-commutativity of quantum theory suggests that generic transformations between quantum systems of references will not be of the quantum controlled form. We will now work out two examples of BQC permutations used as a 'genuinely quantum' reference frame transformation: (A) in first-quantization formalism, demonstrating they can be used to control on non–commuting observables at different locations, and (B) in second-quantization formalism, demonstrating that they can localize states that can not be localized with QC transformations.

A. BQC frames in quantum mechanics

Recall the example discussed in Section II A, of a particle on a line. The relative observable was the (signed) distance $\hat{X}_1$ ($\hat{X}_2$) as seen by O_1 (O_2). The QP that changes from the frame of O_1 to the frame of O_2 is

$$u = \sum_x T_{-x,S} \otimes |x\rangle \langle x|_{O_2} \tag{35}$$
$$\hat{X}_2 = \hat{X}_1 - \hat{X}_{O_2} = u\, \hat{X}_1\, u^\dagger,$$

with $T_{-x}|y\rangle_S = |y-x\rangle_S$ translating the $\hat{X}_1$ eigenbasis.

In order for $\hat{X}_2$ to make sense as a relative to O_2 observable, we had assumed it only acts non-trivially on the state of O_2. More precisely, we had assumed that for every x_1 there exists an observable $\hat{X}_{2,x_1}$ such that $\hat{X}_2 |x_1\rangle |\phi\rangle = |x_1\rangle \otimes \hat{X}_{2,x_1} |\phi\rangle$ for all $|\phi\rangle \in \mathcal{H}_{O_2}$

The difference between QC and BQC transformations is directly linked to the commutativity between the operators $\hat{X}_{2,x_1}$ for all x_1. When they all commute, their projections onto their eigenspaces $u_{x_1x_2}$ will also all commute, and the QP that results will be QC. When the $\hat{X}_{2,x_1}$ do not all commute, it will be BQC. In the example worked out in Section II A and recalled above, the $\hat{X}_{2,x_1}$ commute for different x_1, since $\hat{X}_{2,x_1} = x_1 \mathbb{1}_{O_2} - \hat{X}_{O_2}$. An example of a relative observable arrived at by a BQC permutation is when the observer's position is used to transform *some* $\hat{X}_1$ (eigen)values, while the observer's *momentum* is used for other (eigen)values of $\hat{X}_1$. Let us now work this out explicitly.

Controlling on two complementary variables

First, recall that the typical QRF transformations found in the literature are quantum controlled (see Section VIII). In particular, this includes the case when a symmetry is assumed corresponding to a group that is the direct product of two or more other groups, and of which the corresponding observables commute. In such cases, the corresponding Hilbert space can be broken down to a direct product of two spaces, each corresponding to a distinct symmetry: the resulting QRF transformation effectively acts on each space separately.

Our task here is to construct, instead, *a QRF transformation which quantum controls on two subspaces corresponding to non-commuting observables, such as momentum and position.* In this case, the Hilbert space cannot be broken down to a product corresponding to the different symmetries, as the corresponding relational observables will not commute, and it will correspond to a BQC permutation.

Accordingly, we will say that two groups G_1, G_2 are *incompatible* if the orthonormal bases created by their (left regular, see previous Section) representations $\{|g_1\rangle\}_{g_1\in G_1}$ and $\{|g_2\rangle\}_{g_2\in G_2}$ *do not commute.* That is, when at least for one pair g_1, g_2 it holds that $|g_1\rangle\langle g_1|$ and $|g_2\rangle\langle g_2|$ do not commute.

For any single group G, all the projectors $|g\rangle\langle g|$ commute by construction. Thus, *it is not possible to control on two incompatible groups with a QC permutation.* The non–commuting projections will directly enter the definition of the quantum permutation which encodes the transformation. Since non–commuting entries are present, *in order to control on two incompatible groups, the only option is to quantum control on subspaces of the system of interest through a BQC permutation.*

Consider that we have two incompatible symmetry groups G_1, G_2 e.g. translations in position and boosts in momentum. A BQC permutation that is block diagonal — of the general form (24) — could be written as

$$u = \sum_{g_1\in G_1} u_{g_1} \otimes |g_1\rangle\langle g_1| + \sum_{g_2\in G_2} v_{g_2} \otimes |g_2\rangle\langle g_2|, \tag{36}$$

where $u_{g_1} v_{g_2}^\dagger = 0$, $u_{g_1} u_{g_1}^\dagger = I_1$, $v_{g_2} v_{g_2}^\dagger = I_2$, for all $g_1 \in G_1, g_2 \in G_2$, with some operators $I_1 + I_2 = \mathbb{1}$. Note that the above equation is in the form of (24) with two blocks, corresponding to the two groups.

Let us now take that O_1, O_2 and S are particles on a line with position and momentum operator. We denote $\hat{X}_{O_2}, \hat{X}_S$ and $\hat{P}_{O_2}, \hat{P}_S$ the position and momentum of O_2 and S respectively relative to O_1. Assume that S and O_2 interact through a Hamiltonian that depends on the absolute ratio of their positions if S is positioned at or to the right of $x = 0$, while it depends on the absolute product of S's position and O_2's momentum otherwise:

$$H = \pi_{\hat{X}_S \geq 0} V\left(x_0 \left|\frac{\hat{X}_S}{\hat{X}_{O_2}}\right|\right) + \pi_{\hat{X}_S<0} V\left(-\frac{x_0}{\hbar}|\hat{X}_S \hat{P}_{O_2}|\right), \tag{37}$$

with π_C a projector on all states that fulfill the condition C. Then, we can write a quantum permutation u that transforms this state from O_1's perspective to O_2's perspective:

$$\begin{aligned} u &= \sum_x u_x \otimes |x\rangle\langle x| + \sum_p v_p \otimes |p\rangle\langle p|, \qquad (38)\\ u_x &= \sum_{x_S \geq 0} |\, x_0\, |x_S/x|\,\rangle \langle x_S|, \\ v_p &= \sum_{x_S<0} |\, -x_0 |x_S p|/\hbar\,\rangle \langle x_S|, \end{aligned}$$

This is a BQC QRF transformation that depends on the non-commuting observables $\hat{X}_{O_2}, \hat{P}_{O_2}$. It is a symmetry of Hamiltonians of the form (37). Note that above we left implicit the state of O_2, which after the transformation is fixed to $|x_{O_2} = x_0\rangle$ for $x_S \geq 0$ and to $|p_{O_2} = \hbar/x_0\rangle$ for $x_S < 0$. Indeed, applying the above u on a basis we get

$$\begin{aligned} u\,|x_S\rangle\,|x\rangle &= |\, x_0\,|x_S/x|\,\rangle\,|x\rangle \qquad \text{if } x_S \geq 0 \qquad (39)\\ u\,|x_S\rangle\,|p\rangle &= |\, -x_0 |x_S p|/\hbar\,\rangle\,|p\rangle \qquad \text{if } x_S < 0. \end{aligned}$$

Here, we see an example of the fact that BQC permutations allow to control on subspaces defined by different locations using arbitrary choices of bases (see (25) and surrounding discussion). The above transformation uses the momentum and position basis, which do not commute, on different parts of the real line.

B. BQC frames in quantum fields

In Section II B, we saw that the QP (7) transforming between the two reference systems was of the form

$$u_{x_2 x_1} = \mathbb{1}_S \otimes \sum_q P_1^{q x_1} \otimes P_2^{q x_2}.$$

where the $P_i^{q x_i}$ are orthogonal projectors on the eigenspace of the reference field $\hat{\chi}_i(q)$ with eigenvalue x^μ (it assigns the x^μ coordinate to the abstract point q). We remarked that this QP will generally be BQC, since we generally expect that for quantum fields there exist some q, q' such that $[\hat{\chi}_i(q), \hat{\chi}_i(q')] \neq 0$ (known as the principle of microcausality). This implies that $[P_i^{q x_i}, P_i^{q' x_i'}] \neq 0$ for some x_i, x_i' and therefore $[u_{x_2 x_1}, u_{x_2' x_1'}] \neq 0$. Strikingly, this implies that the non-commutativity of fields in quantum field theory is therefore directly linked to 'genuinely quantum' reference frames. Let us now work out an example of what can be achieved with a BQC QRF and which is not possible with a QC QRF.

Localizing two particles simultaneously

We now use the discrete quantum fields of Section II B to give an example of a BQC permutation that transforms a bipartite state in which both particles are initially in indefinite states, to a state frame in which both particles become definite. We then show that it is impossible to transform this state to a quantum reference frame in which both particles become definite using a quantum controlled transformation.

Assume two massive particles A, B described by quantum scalar fields $\hat{\varphi}_A, \hat{\varphi}_B$ with Fock spaces $\mathcal{H}_A, \mathcal{H}_B$ and two reference fields $\hat{\chi}_1, \hat{\chi}_2$ with Fock spaces $\mathcal{H}_{O_1}, \mathcal{H}_{O_2}$. The total space is then $\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_{O_1} \otimes \mathcal{H}_{O_2}$. We assume the particles A and B to be masses with non-relativistic motion, in the sense that $mc\Delta x \gg \hbar$, where Δx is the scale of difference between eigenvalues of the reference fields $\hat{\chi}_{1,2}$. The fields $\hat{\chi}_1, \hat{\chi}_2$ satisfy the conditions set out in Section II B in order to be good reference fields, and are otherwise arbitrary.

Assume further that the reference field O_1 is "classical", or – more precisely – definite. This means that there is exactly one value $x_q \in \mathbb{R}^4$ for every $q \in \mathcal{M}$: $\hat{\chi}_1(q) = x_q \mathbb{1}_{O_1}$. In terms of the projectors, $P_1^{qx} = \delta_{x x_q} \mathbb{1}_{O_1}$.

Now, consider the state $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_{O_1} \otimes \mathcal{H}_{O_2}$

$$\begin{aligned} |\psi\rangle =& \frac{1}{\sqrt{2}} |x_A\rangle^{O_1} \frac{|x_B\rangle^{O_1} + |x_B'\rangle^{O_1}}{\sqrt{2}} |\phi\rangle_{O_1 O_2} + \qquad (40)\\ & \frac{1}{\sqrt{2}} |x_A'\rangle^{O_1} \frac{|x_B\rangle^{O_1} - |x_B'\rangle^{O_1}}{\sqrt{2}} |\phi^\perp\rangle_{O_1 O_2}. \end{aligned}$$

Here, the $x_A, x_A', x_B, x_B' \in \mathbb{R}^4$ are all different, $|\phi\rangle, |\phi^\perp\rangle \in \mathcal{H}_{O_1} \otimes \mathcal{H}_{O_2}$ are any two orthogonal states of the joint system of reference fields and $|\phi^\pm\rangle = \frac{1}{\sqrt{2}}\left(|\phi\rangle \pm |\phi^\perp\rangle\right)$. A state $|y\rangle^{O_1}$ describes a particle being created at the specified spacetime point relative to O_1:

$$\begin{aligned} |y_A\rangle^{O_1} &:= a_1^\dagger(y_A)\,|\Omega_A\rangle = a^\dagger(q_A)\,|\Omega_A\rangle \qquad (41)\\ |y_B\rangle^{O_1} &:= b_1^\dagger(y_B)\,|\Omega_B\rangle = b^\dagger(q_B)\,|\Omega_B\rangle \end{aligned}$$

for any $y_{A.B}, q_{A,B}$ such that $P_1^{q_A y_A} = P_1^{q_B y_B} = \mathbb{1}_{O_1}$. The states $|\Omega_A\rangle \in \mathcal{H}_A, |\Omega_B\rangle \in \mathcal{H}_B$ are the vacuum states of the A and B fields. The annihilation operators for A, B were denoted a, b and a subscript 1 denotes the field relative to O_1 as defined in (4).

We now ask whether there exists a transformation, for which relative to O_2 the state is

$$|\psi\rangle = a_2^\dagger(x_A) b_2^\dagger(x_B)\ |\Omega\rangle_A |\Omega\rangle_B |\phi\rangle_{O_1 O_2}. \qquad (42)$$

That is, so that *both* particles are localised. We show in Appendix B that this transformation from O_1 to an O_2 indeed exists. Using the notation of Section II B, in particular compare to (7), the BQC quantum reference frame which achieves the double localization is given as follows

$$\begin{aligned}
u_{x_A x_A} &= \mathbb{1}_{AB} \otimes \sum_q P_1^{q x_A} \otimes P_2^{q x_A} = |\phi\rangle\langle\phi| \, x \\
u_{x_A x'_A} &= \mathbb{1}_{AB} \otimes \sum_q P_1^{q x_A} \otimes P_2^{q x'_A} = |\phi^\perp\rangle\langle\phi^\perp| \\
u_{x'_A x_A} &= \mathbb{1}_{AB} \otimes \sum_q P_1^{q x'_A} \otimes P_2^{q x_A} = |\phi^\perp\rangle\langle\phi^\perp| \\
u_{x'_A x'_A} &= \mathbb{1}_{AB} \otimes \sum_q P_1^{q x'_A} \otimes P_2^{q x'_A} = |\phi\rangle\langle\phi| \\
u_{x_B x_B} &= \mathbb{1}_{AB} \otimes \sum_q P_1^{q x_B} \otimes P_2^{q x_B} = |\phi^+\rangle\langle\phi^+| \\
u_{x_B x'_B} &= \mathbb{1}_{AB} \otimes \sum_q P_1^{q x_B} \otimes P_2^{q x'_B} = |\phi^-\rangle\langle\phi^-| \\
u_{x'_B x_B} &= \mathbb{1}_{AB} \otimes \sum_q P_1^{q x'_B} \otimes P_2^{q x_B} = |\phi^-\rangle\langle\phi^-| \\
u_{x'_B x'_B} &= \mathbb{1}_{AB} \otimes \sum_q P_1^{q x'_B} \otimes P_2^{q x'_B} = |\phi^+\rangle\langle\phi^+|. \qquad (43)
\end{aligned}$$

Note that this is the quantum permutation (21) with $\pi_1 = |\phi\rangle\langle\phi|, \pi_2 = |\phi^+\rangle\langle\phi^+|$. P_i^{qx} is the projection on the x eigenspace of $\chi_i(q)$.

In the resulting quantum coordinate system (as defined by the reference test field $\hat{\chi}_2$), both particles are created in definite spacetime points. This is not possible to achieve with a quantum controlled permutation. The proof, given in Appendix B, is a formalization of the following observations. The state $|\psi\rangle$ is not a superposition of definite bipartite states: for $|\tilde{\phi}\rangle \in \mathcal{H}_{O_1} \otimes \mathcal{H}_{O_2}$, there does not exist a state $\langle\tilde{\phi}|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ that has both particles in definite states. In other words, $\langle\tilde{\phi}|\psi\rangle \neq |\tilde{x}_A\rangle^{O_1} |\tilde{x}_B\rangle^{O_1}$ for any $\tilde{x}_A, \tilde{x}_B$. Therefore, a QC permutation, which is a superposition of definite permutations, cannot render the state definite.

Let us now comment on how the above are linked to the observation made in Section VI that BQC permutations can be understood as making a local choice of control basis for the observers sector $\mathcal{H}_{O_1} \otimes \mathcal{H}_{O_2}$ at each point. We had seen that quantum permutations can be written in the form (25), repeated here for convenience

$$u = \sum_x^N \sum_{k=1}^{K_x} |y_k(x)\rangle\langle x| \otimes \pi_k^{(x)}.$$

In the example we studied above, the new reference field $\hat{\chi}_2$ does not commute with itself at the points q_A and q_B (for instance), which correspond to the coordinates x_A and x_B relative to O_1 (assigned by the $\hat{\chi}_1$ reference field). That is, $[\hat{\chi}_2(q_A), \hat{\chi}_2(q_B)] \neq 0$.

The above example demonstrates that there exist states (such as (40)), which in order to localize by an appropriate choice of frame, the reference field will not commute with itself at some points. *Therefore, the fact that quantum fields do not commute at different points compels the extension of the QRF framework to include BQC transformations.*

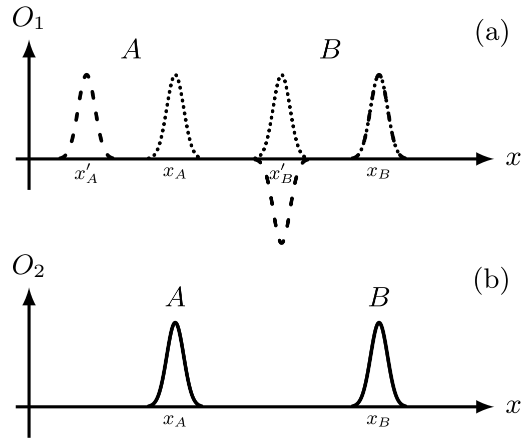

Figure 4. (a) The state (40) of the two particles with respect to observer O_1 is a superposition of two terms: a term where A is localized at x_A and B is in a superposition of locations x_B and x'_B, and a term where A is localized at x'_A and B is in a superposition of the same two locations but now with a relative minus sign. Here, the different line patterns represent the entanglement across branches. In the O_1 frame the particles are also entangled with the reference fields, which are in the states $|\phi\rangle_{O_1 O_2}$ and $|\phi^\perp\rangle_{O_1 O_2}$ in the respective branches. These field states are not shown in the figure. (b) From the perspective of O_2, the same state is (42), with A and B at the definite locations x_A and x_B. In contrast to the frame of O_1, in the frame of O_2 there is no entanglement, neither among the particles nor between the particles and the fields.

C. Brief comments on relevant QRF literature

We pause to make some comments concerning some particularly relevant QRF literature.

A QRF transformation that achieves a similar goal to that presented in Section IX A appeared in [29]. In that work, an additional degree of freedom was introduced in order to control on the bases $|x\rangle$ and $|p\rangle$. In the notation of that work it is a variable θ. A basis of the joint system is constructed of the form $\{|x\rangle|\theta\rangle\}_{\theta\in\Theta} \cup \{|p\rangle|\theta\rangle\}_{\theta\notin\Theta}$, with Θ some subset of possible θ values. This allows to use a QC transformation in order to control on both $|x\rangle$ and $|p\rangle$. This is different than what we have done, as we did not introduce a new degree of freedom. Rather, we gave up on the restriction to QC transformations.

It is also interesting to briefly discuss the case of non-ideal QRFs [8–10]. The QRF transformations we have discussed in this manuscript, are known as ideal QRFs. This is because it is assumed that different states of an observer (corresponding to different group elements) are perfectly distinguishable (orthogonal). Non-ideal QRFs relax this restriction, and by virtue of that may allow for simultaneous control of non-commuting observables (although, we are not aware of a work demonstrating this). The interpretation of quantum permutations that we have presented is as ideal quantum reference frames. That is, we have assumed that any two O_2 states that disagree on the new value $y \neq y'$ of a point x relative to O_1 are orthogonal, i.e. $u_{xy}u_{xy'} = 0$. The fact that they allow simultaneous control on complementary variables is not due to the introduction of an uncertainty threshold for the observer's state (which is the characteristic of non-ideal QRFs), but rather in not requiring that the basis for $\mathcal{H}_O$ is chosen globally. As we have seen, BQC permutations are constructed by making an arbitrary choice of basis for $\mathcal{H}_O$ at each location.

Turning to the second example presented in Section IX B, this was inspired by the work [16], which used QC transformations to define quantum coordinate changes defined through reference fields. Section IX B (and Section II B) extend what was done in [16] to incorporate the fact that quantum fields do not commute at different points. Because BQC permutations allow the control basis for the transformation to be different at every point, the transformation is not definite or 'classical-like' for any post-selection on $\mathcal{H}_O$ (meaning, after projecting on an observer's state). This is precisely what allows the transformation to not preserve 'relative definiteness', namely localizing B while not delocalizing A, even after post selection on $|\phi\rangle$ or $|\phi^\perp\rangle$ (see end of Section IX B).

Let us see why a QC permutation, such as those discussed in [16], does not permit for localization of A and B. In simplified notation, the QC transformation would act on some state $\sum_\chi \sigma_\chi \otimes |\chi\rangle\langle\chi|\Psi\rangle$, with σ_χ describing a definite transformation of A, B and $|\chi\rangle$ the observer states. Post-selecting on a basis element $|\chi_*\rangle$, the picture becomes definite or classical-like, that is, $|\chi_*\rangle\langle\chi_*|\sum_\chi \sigma_\chi \otimes |\chi\rangle\langle\chi|\Psi\rangle = \sigma_{\chi_*} \otimes |\chi_*\rangle\langle\chi_*|\Psi\rangle$. Therefore, for every state $|\chi_*\rangle$ of the observer, the relative definiteness of A, B cannot be changed, as classical permutations take definite locations to definite locations and superpositions to superpositions. Intuitively, because BQC permutations do not restrict to a branch-by-branch classicality of the transformation, they are able to change the relative definiteness.

In summary, beyond quantum controlled permutations introduce a dependence of the control basis on x values, which allows for more general interplay between the quantum coordinate system and the quantum objects which it describes.

X. GENUINELY QUANTUM SYMMETRIES OF THE ISING MODEL

In this section we will see how the formalism we have developed, whereby QPs are understood as quantum reference system transformations, can be used to demonstrate new symmetries of the Ising model.

The Ising model is a system of spins on the nodes of a graph Γ such that only neighboring spins directly interact. In addition, an external magnetic field affects all spins. The Hamiltonian is

$$H = -J \sum_{x \sim_\Gamma x'} \hat{\sigma}_1(x)\hat{\sigma}_1(x') - h \sum_x \hat{\sigma}_1(x), \tag{44}$$

with constants $J, h \geq 0$ representing the strength of interaction and the magnetic field respectively. The operator $\hat{\sigma}_1(x) = |\uparrow\rangle\langle\uparrow|_{1,x} - |\downarrow\rangle\langle\downarrow|_{1,x}$ is the spin operator on $\mathcal{H}_{1,x}$, which represents the spin on the node of the graph that is at $x \in \mathbb{R}^n$. The state space is $\mathcal{H}_S \simeq \bigotimes_x \mathcal{H}_{1,x}$, with an orthonormal basis $|s\rangle = \bigotimes_x |s_x\rangle$, $s_x = \uparrow, \downarrow$.

Below, we combine the the second quantization formalism presented in Section II B and IX B, and the first quantization toy-model of a spin on a graph of Section VII. We interpret x to be labels of the nodes of the graph, given according to some reference. To transform $\sigma_1(x)$ to the reference frame of O_2, we simply write

$$\hat{\sigma}_2(y) = \sum_x \hat{\sigma}_1(x) u_{yx} \tag{45}$$

as was found in Section II B, see (6). The Hamiltonian from O_2's reference then looks like:

$$H = -J \sum_{y,y'} \hat{\sigma}_2(y) A_{2,yy'} \hat{\sigma}_2(y') - h \sum_y \hat{\sigma}_2(y), \tag{46}$$

with $A_{2,yy'} = \sum_{x,x'} A_{1,xx'} u_{xy} u_{x'y'} = (u^\dagger A_1 u)_{yy'}$ and A_1 the adjacency matrix of the graph, that is $A_{1,xx'} = 1$ if $x \sim_\Gamma x'$ and else $A_{1,xx'} = 0$. Symmetries of the Ising model are reference system transformations u such that the Hamiltonian retains its form – or in other words, such that the equations of motion do not change. This only happens if $u\, A_1\, u^\dagger = A_1$, which is exactly the definition of quantum automorphisms of the graph.

The usual symmetries of the Ising model are classical permutations u that are automorphisms of Γ. As discussed in Section VII, if the graph has more than one classical symmetries, QC permutations that are quantum superpositions of those will also be symmetries of the Ising model. If the graph has genuinely quantum symmetries, this corresponds to the existence of BQC permutations that are quantum automorphisms of Γ.

Let us see a concrete example. Consider the case of the graph Γ depicted in Fig. 5. A quantum symmetry for the Ising model on this graph is achieved by moving to some other reference O_2 such that the corresponding u from (7) is a quantum symmetry of this graph. An

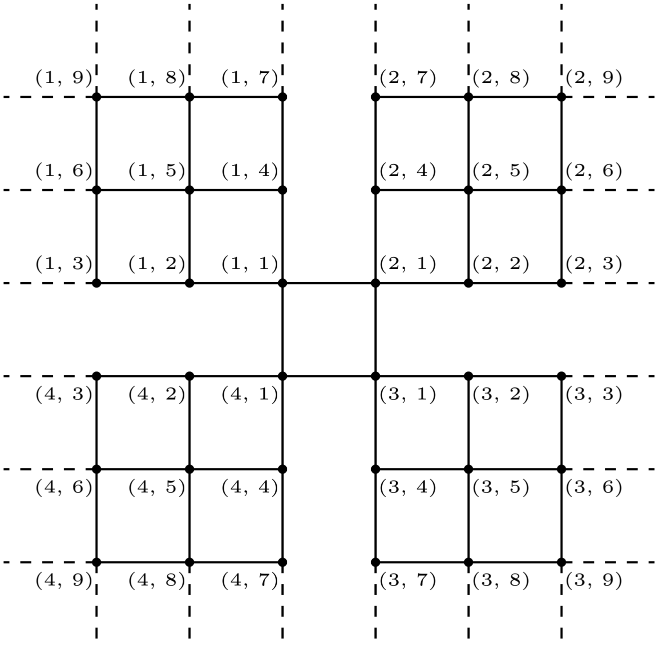

Figure 5. An example of a 'large' graph whose corresponding Ising model is symmetrical under the quantum reference system transformation (47). This was constructed by extending the genuinely quantum symmetry of the 4-cycle discussed in Section VII. As far as we are aware, it is not known whether regular lattices (such as a square lattices) possess genuinely quantum symmetries, as this is not trivial to check.

example for such a u is

$$u_{yx} = \begin{cases} \pi_1 & x = y = (1,j) \text{ or} \\ & x = y = (3,j) \\ \mathbb{1} - \pi_1 & x = (1,j), y = (3,j) \text{ or} \\ & y = (1,j), x = (3,j) \\ \pi_2 & x = y = (2,j) \text{ or} \\ & x = y = (4,j) \\ \mathbb{1} - \pi_2 & x = (2,j), y = (4,j) \text{ or} \\ & y = (2,j), x = (4,j) \\ 0 & \text{else} \end{cases} \tag{47}$$

with π_1, π_2 two non-commuting orthogonal projections on $\mathcal{H}_{O_1} \otimes \mathcal{H}_{O_2}$, which can be constructed by taking any two orthogonal states in that space $\langle \phi | \phi^\perp \rangle = 0$ and setting $|\pm\rangle = \frac{1}{\sqrt{2}} \left(|\phi\rangle \pm |\phi^\perp\rangle \right)$, $\pi_1 = |\phi\rangle \langle \phi|, \pi_2 = |+\rangle \langle +|$.

Note that the same holds if one "closes" the graph by adding the the edges $(1, n^2) \leftrightarrow (2, n^2)$, $(2, n^2) \leftrightarrow (3, n^2)$, $(3, n^2) \leftrightarrow (4, n^2)$, $(4, n^2) \leftrightarrow (1, n^2)$, with n^2 being the number of nodes at every quarter of the graph (9 of which are drawn in Fig. 5). This would make the model admit cyclic boundary conditions.

In the above example, we used a rather ad-hoc large graph, which we constructed by extending the genuinely quantum symmetry of the 4-cycle discussed in Section VII, see Figure 5. It should be noted that there are theorems that show that almost all general graphs do not have quantum symmetries [30], while almost all tress do [31]. This is situation is similar as for classical symmetries [32]. However, as far as we are aware, it is not known whether regular lattices possess genuinely quantum symmetries, and this is difficult to check for any given large graph. On the other hand, trees typically arise when causal structure is present.

On this note, it is interesting to discuss what is the physical interpretation of the reference frame that corresponds to genuinely quantum graph symmetry, according to the results in the preceding Sections. In order to allow a quantum system as the reference, with respect to which the node labels are defined, we need to include it in the description. Recall that $\mathcal{H}_{O_1}, \mathcal{H}_{O_2}$ are the Hilbert spaces of two quantum reference fields $\hat{\chi}_1(q), \hat{\chi}_2(q)$ defined on points in some abstract set $q \in \mathcal{M}$. The Hilbert space is therefore $\mathcal{H}_S \otimes \mathcal{H}_{O_1} \otimes \mathcal{H}_{O_2}$. The reference used to write the Hamiltonian (44) is O_1.

The classical and QC symmetries are physically interpreted with respect to commuting reference fields. That is, if we use two reference fields $\hat{\chi}_1, \hat{\chi}_2$ to name the nodes, which are such that $[\hat{\chi}_i(q), \hat{\chi}_i(q')] = 0$ for all q, q', the Hamiltonian stays invariant under the QP u that relates them, if and only if u is a classical automorphism of the graph or a QC permutation composed from classical automorphisms.

Physically realizing a frame that corresponds to a genuinely quantum graph symmetry, would require that the reference fields used to label the nodes are such that $[\hat{\chi}_i(q), \hat{\chi}_i(q')] \neq 0$ for some q and q', so that the u that relates them can be a quantum automorphism of the graph, that is, a BQC permutation. This will generally arise for fields that live in Lorentzian signature spacetimes. Let us now turn to such an example.

XI. QUANTUM SCALAR FIELD ON CURVED SPACETIME

We now study a discretized model for a scalar field $\hat{\varphi}$ on curved spacetime, and show that it is invariant under quantum permutations that obey a discrete differentiability condition. The continuum action of a scalar field on a curved spacetime is

$$S = \int \sqrt{-g(x)} \, \mathrm{d}^4 x \, \frac{1}{2} \partial^\mu \hat{\varphi}(x) \, g_{\mu\nu}(x) \, \partial^\nu \hat{\varphi}(x), \tag{48}$$

where $g_{\mu\nu}$ is the spacetime metric and g its determinant. This action is invariant under diffeomorphisms.

Using similar notation as in Sections II B and IX B, let us define the discrete action from the perspective of O_1

$$S = \sum_{x, x_u, x_v} \Delta\hat{\varphi}_1(x, x_u) \, \hat{g}_1(x, x_u, x_v) \, \Delta\hat{\varphi}_1(x, x_v) \tag{49}$$

where recall that the subscript 1 signifies fields as seen by O_1. The discrete field derivative is defined as

$$\Delta\hat{\varphi}_1(x, x_u) := \frac{\hat{\varphi}_1(x_u) - \hat{\varphi}_1(x)}{\|x_u - x\|}. \quad (50)$$

where $\|\cdot\|$ could be any norm on $\mathbb{R}^n$. The operator $\hat{g}_1(x, x_u, x_v)$ on $\mathcal{H}_S \otimes \mathcal{H}_{O_1}$ is a discrete and quantum metric. We take $\hat{g}_1$ to be a field (a hermitian operator at every point) that is symmetric under the exchange of x_u and x_v, as in the classical case. Namely, $g_1(x, x_u, x_v) = g_1(x, x_v, x_u) = g_1(x, x_u, x_v)^\dagger$.

We assume that $\hat{g}_1(x, x_u, x_v)$, commutes with u_{xy} for all x, y. This assumption is motivated from the examples studied in Section II B, Section IX B and Section X, where we saw that the fields $\hat{\varphi}_1(x)$ and $\hat{\varphi}_2(x)$ commute with u_{xy} for all x, y. This can be seen from (5) and (7), see also see (45). Intuitively, for every $\hat{g}_1(x, x_u, x_v) \neq 0$, the points x_u, x_v could be both thought of as "close" to x, or as neighbors of it on some graph or lattice.

We now move to a different coordinate system $\hat{\chi}_2$. The corresponding QP u is given by (7), repeated here for convenience

$$u_{x_2x_1} = \mathbb{1}_S \otimes \sum_q P_1^{qx_1} \otimes P_2^{qx_2}. \quad (51)$$

We assume that $u_{x_2x_1}$ obeys the following *discrete differentiability* condition: given x, x_u such that $\hat{g}_1(x, x_u, x_v) \neq 0$ at least for one x_v, it holds that $u_{xy}u_{x_uy_a} = 0$ unless y_a is of a specific value, denoted by $D_{xy}(x_u)$. This is assumed to be an invertible function, i.e. $D_{xy}(x_u) \neq D_{xy}(x'_u)$ for any $x_u \neq x'_u$. Only some QPs will satisfy this condition, which we call differentiable.

The transformation rule for the field is given by (6), repeated here for convenience

$$\hat{\varphi}_1(x) = \sum_y u_{xy}\, \hat{\varphi}_2(y). \quad (52)$$

The discrete field derivative transforms as

$$\Delta\hat{\varphi}_1(x, x_u) = \sum_y u_{xy}\, J_{xy}(x_u)\, \Delta\hat{\varphi}_2\big(y, D_{xy}(x_u)\big) \quad (53)$$

where we have defined the (discrete) Jacobian

$$J_{xy}(x_u) := \frac{\|D_{xy}(x_u) - y\|}{\|x_u - x\|}. \quad (54)$$

We prove in Appendix C that the action retains its form when moving to the perspective of O_2. That is,

$$S = \sum_{y, y_a, y_b} \Delta\hat{\varphi}_2(y, y_a)\, \hat{g}_2(y, y_a, y_b)\, \Delta\hat{\varphi}_2(y, y_b), \quad (55)$$

with

$$\hat{g}_2(y, y_a, y_b) := \sum_x u_{xy}\, J_{xy}\big(D_{xy}^{-1}(y_a)\big)\, J_{xy}\big(D_{xy}^{-1}(y_b)\big) \quad (56)$$
$$\hat{g}_1\big(x, D_{xy}^{-1}(y_a), D_{xy}^{-1}(y_b)\big),$$

In Appendix C we also give (i) a concrete example of such a symmetry of the action, namely a metric and a transformation obeying the differentiability condition (ii) an example of a metric and a transformation that do not obey the differentiability condition, and show that in this case the action does not remain invariant.[9]

Let us summarize. We made a heuristic attempt at a definition for differentiability for quantum permutations: given that u sends x to y, D_{xy} dictates that a nearby point x_u will be sent to $y_a = D_{xy}(x_u)$. We saw that the discrete analogue of the quantum field derivative and quantum metric transform analogously to the way tensors transform in general relativity. The invariance of the action then follows.

Therefore, we have shown that there are QPs that satisfy a discrete sense of differentiability and which correspond to new, genuinely quantum, symmetries of a background independent action.

XII. DISCUSSION

We have demonstrated that quantum permutations naturally arise in quantum theory as generic quantum reference frame transformations. Importantly, (10) implies that the non-commutativity of quantum fields at different locations gives rise precisely to the condition that the frame transformation must be a genuinely quantum permutation.

With this observation as the point of departure, we set out to investigate how genuine QPs non-trivially extend the framework of (ideal) quantum reference frames. We showed that the special subclass of QPs that correspond to superposing classical graph isomorphisms are quantum controlled permutations. Therefore, the genuinely quantum permutations are those that cannot be cast in the form (19), where the observer's space serves the role of a quantum control space. Genuinely quantum or BQC permutations are interesting because they can encode symmetries present in quantum theory with no classical counterpart: there are classically not-isomorphic graphs that are mapped to each through a BQC permutation (they are quantum isomorphic). Here, we studied quantum permutations in a quantum information language which has allowed us to study the role of this notion of quantum symmetry in a more general context than graphs, e.g by taking the quantum permutation to act on the eigenbasis elements of observables.

We observed that QC permutations correspond to making a global choice of control basis, the same at each location, while BQC permutations correspond to choosing one control basis per location and so that these bases will not commute on (at least two) different locations

[9] Interestingly, the differentiability condition above is fulfilled by any classical permutation, while this is not the case for quantum permutations.

(see (25) and the discussion that follows). In this sense, BQC permutations incorporate the non-commutativity of quantum theory as the non-commutativity of the local choice of quantum reference frame.

Leveraging this insight, we produced examples of genuinely quantum reference frame transformations. We first demonstrated that QC permutations can be understood as symmetries of the dynamics of a physical system, in particular of any Hamiltonian that is symmetric under (ideal) quantum reference frame transformations studied in the literature. We gave a Hamiltonian that is symmetric under a BQC unitary which 'packages together' transformations that control on non-commuting observables on different parts of the real line. We then saw how to generalise the notion of quantum coordinates that appeared in [16] to BQC transformations, demonstrating how to localize the state (40), which cannot be localized with QC transformations.

Putting the above together, we saw that the Ising Hamiltonian, when defined on a graph with a genuinely quantum symmetry, is invariant under the corresponding BQC permutation—and that the physical realization of such a frame choice will involve the use of a reference quantum field that does not commute on some of the graph nodes.

The non-commutativity of field operators is usually related to causality: the principle of microcausality is the fact that quantum fields commute at spacelike separation but in general do not at timelike separation. Given that almost all trees (graphs with a partial order) have genuinely quantum symmetries [31], this implies that these new symmetries may become particularly important in a relativistic context of theories defined on spacetimes with Lorentzian signature. Accordingly, we studied BQC permutations that fulfill a discrete differentiability condition, and showed that, remarkably, they leave invariant a discretization of the action of a scalar field on a curved spacetime, see (55). This strongly indicates that *a notion of continuous quantum permutations would provide a definition for 'genuinely quantum diffeomorphisms'*; quantum diffeomorphisms that are transformations more general than superpositions of classical diffeomorphisms. Our results set the stage to consider the continuum case, which will be the topic of a future work.

Recently, prototypical invariants under quantum coordinates changes have begun to emerge and their meaning investigated. In [26], such an invariant was used to give the first general relativistic definition for indefinite causal ordering (ICO), applicable both when ICO is implemented on an optical bench and when ICO takes place due to a quantum superposition of gravitational fields. General knot invariants in the context of ICO have since been found [33]. Another example is the demonstration that while entanglement and coherence are not generically preserved under quantum (controlled) reference frame transformations, their sum is preserved [28]. The idea of quantum coordinate changes was used in [34] to argue that when entanglement generated through the gravitational interaction, the proper distance must be set in superposition (because it can not be made definite with a quantum coordinate change). Most closely related to this work, it has recently been shown that the invariance under quantum controlled permutations rules out parastatistics [35] (providing a 'reason' for why we see only bosons and fermions). This is an intriguing demonstration of how invariance under quantum reference frames can be used as a guiding principle for physics.

All of the above works concern a notion of invariance under QC permutations. Furthermore, typically an explicit fixed choice of basis is made (QC transformations for a fixed choice of basis form a group). It is intriguing to ask how the picture may change when considering instead invariance under general BQC transformations. Intuitively, it would seem that a *smaller* set of invariants would survive if invariance under genuinely quantum reference frames is posited (instead of invariance under the QC kind). The main challenge here is that the quantum permutations are a sort of quantum group: the group closure of QPs is contained in the group of unitaries of the same dimensionality, whose blocks sum up to one over rows and columns. Whether this closure is the entire group of unitaries with unit sum of rows and columns is unknown, but it is easy to show that it cannot be the entire unitary group. One possibility is to forgo a group structure and use instead a weaker equivalence relation [30], in order to define the quantized version of the 'glocal' observables studied in [36].

Finally, we mention another interesting further direction: to investigate whether the theory of magic squares [37] may allow to extend non-ideal QRFs, similarly to how we have done here for ideal QRFs using QPs (magic unitaries). Magic squares are not unitaries but have positive blocks, so the blocks can be written as weighted sums of projections. This seems to point towards some type of coarse-graining of QP blocks.

In closing, we have seen that the theory of quantum permutations significantly extends the notion of symmetry in physics. We have only begun here to uncover the implications.

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Appendix A: Proof of the appearance of quantum permutations in the quantum mechanical setting

From 1, $\langle x_1| \hat{X}_2 |x_1'\rangle = \delta_{x_1 x_1'} \hat{X}_{2,x_1}$, so $\hat{X}_2 = \sum_{x_1} |x_1\rangle\langle x_1| \otimes \hat{X}_{2,x_1}$.

Any $\hat{X}_{2,x_1}$ is hermitian, thus can be written as $\hat{X}_{2,x_1} = \sum_{x_2} x_2 \; u_{x_1x_2}$, where the operators $u_{x_1x_2} = (u_{x_1x_2})^\dagger = (u_{x_1x_2})^2$ are orthogonal projections that satisfy $\sum_{x_2} u_{x_1x_2} = \mathbb{1}_{O_2}$.

From 2 it follows that the x_2 eigenspace of $\hat{X}_{2,x_1}$ and the x_2 eigenspace of $\hat{X}_{2,x_1'}$ for $x_1 \neq x_1'$ are orthogonal. Therefore $u_{x_1x_2}u_{x_1'x_2} = \delta_{x_1x_1'}\, u_{x_1x_2}$. Denote now $u_{x_2} := \sum_{x_1} u_{x_1x_2}$ and $\mathrm{r}(\cdot)$ the rank of a projection. From the orthogonality we have just obtained, $\mathrm{r}(u_{x_2}) = \sum_{x_1} \mathrm{r}(u_{x_1x_2})$. Summing over x_2 one gets $\sum_{x_2} \mathrm{r}(u_{x_2}) = \sum_{x_1}\sum_{x_2} \mathrm{r}(u_{x_1x_2})$. Since $u_{x_1x_2}$ are orthogonal projections and $\sum_{x_2} u_{x_1x_2} = \mathbb{1}_{O_2}$, it follows that $\sum_{x_2} \mathrm{r}(u_{x_1x_2}) = d$ for all x_1. Thus $\sum_{x_2} \mathrm{r}(u_{x_2}) = \sum_{x_1} d = Nd$. Since the rank of a projection can is maximally the dimension of the space, we have here N numbers $\leq d$ that sum up to Nd. This means that $\mathrm{r}(u_{x_2}) = d$ for all x_2. As the only maximal rank projection is the identity, $u_{x_2} = \sum_{x_1} u_{x_1x_2} = \mathbb{1}_{O_2}$ for all x_2.

Now returning to the relative value observable,

$$\begin{aligned}\hat{X}_2 &= \sum_{x_1} |x_1\rangle\langle x_1| \otimes \hat{X}_{2,x_1} \qquad \text{(A1)}\\ &= \sum_{x_1} |x_1\rangle\langle x_1| \otimes \sum_{x_2} x_2\, u_{x_1x_2}\\ &= \sum_{x_1,x_1',x_2} x_2 \;|x_1\rangle\langle x_1'|_S \otimes u_{x_1x_2}u_{x_1'x_2}\\ &= u\, \hat{X}_1\, u^\dagger,\end{aligned}$$

with

$$u = \sum_{x_1,x_2} |x_1\rangle\langle x_2|_S \otimes u_{x_1x_2}. \qquad \text{(A2)}$$

The opposite direction follows straightforwardly.

Appendix B: Localizing two particles simultaneously

The non-relativistic limit lets us write the annihilation operators approximately as $a_1(x) = \frac{1}{\sqrt{2}}\left(\frac{\sqrt{mc^2}}{\hbar}\,\hat{\varphi}_{A,1}(x) + \frac{i}{\sqrt{mc^2}}\,\hat{\Pi}_{A,1}(x)\right)$ with the conjugate momentum of A relative to O_1 denoted $\hat{\Pi}_{A,1}(x)$, and same for b and B. The reason is that in this regime, the Klein–Gordon equation that A, B satisfy, $\left(\frac{1}{c^2}\partial_t^2 - \nabla^2 + \left(\frac{mc}{\hbar}\right)^2\right)\hat{\varphi}_1(x) = 0$, looks like a family of equations for independent simple harmonic oscillators at every x because the mass term $\left(\frac{mc}{\hbar}\right)^2\hat{\varphi}_1(x)$ dominates the gradient term $\nabla^2\hat{\varphi}_1(x)$. Here, since we work in the discrete, ∂_t and ∇ are understood as their standard finite-difference analogues.

The same transformation rule (6) that we found for a field follows identically for its conjugate momentum. It is thus straightforward to see that it also holds for a and b.

The state can be written as

$$\begin{aligned}|\psi\rangle =& \frac{1}{\sqrt{2}}\, a_1^\dagger(x_A)\, \frac{b_1^\dagger(x_B) + b_1^\dagger(x_B')}{\sqrt{2}}\; |\Omega\rangle_A |\Omega\rangle_B |\phi\rangle_{O_1O_2}\\ &+ \frac{1}{\sqrt{2}}\, a_1^\dagger(x_A')\, \frac{b_1^\dagger(x_B) - b_1^\dagger(x_B')}{\sqrt{2}}\; |\Omega\rangle_A |\Omega\rangle_B |\phi^\perp\rangle_{O_1O_2}.\end{aligned} \qquad \text{(B1)}$$

From the definiteness of O_1, it holds that $a_1(y_A) = a(q_A) \otimes \mathbb{1}_{BO_1O_2}$ and $b_1(y_B) = b(q_B) \otimes \mathbb{1}_{AO_1O_2}$. Hence $a_1^\dagger(\cdot)$ commutes with $b_1^\dagger(\cdot)$, and we may write

$$\begin{aligned}|\psi\rangle =& \frac{b_1^\dagger(x_B) + b_1^\dagger(x_B')}{2}\; a_1^\dagger(x_A)\; |\Omega\rangle_A |\Omega\rangle_B |\phi\rangle_{O_1O_2}\\ &+ \frac{b_1^\dagger(x_B) - b_1^\dagger(x_B')}{2}\; a_1^\dagger(x_A')\; |\Omega\rangle_A |\Omega\rangle_B |\phi^\perp\rangle_{O_1O_2}.\end{aligned} \qquad \text{(B2)}$$

Applying the transformation rule for A,

$$\begin{aligned}a_1^\dagger(x_A) &= a_2^\dagger(x_A)\,|\phi\rangle\langle\phi| + a_2^\dagger(x_A')\,|\phi^\perp\rangle\langle\phi^\perp|,\\ a_1^\dagger(x_A') &= a_2^\dagger(x_A)\,|\phi^\perp\rangle\langle\phi^\perp| + a_2^\dagger(x_A')\,|\phi\rangle\langle\phi|,\end{aligned} \qquad \text{(B3)}$$

So the state is

$$\begin{aligned}|\psi\rangle =& \frac{b_1^\dagger(x_B) + b_1^\dagger(x_B')}{2}\; a_2^\dagger(x_A)\; |\Omega\rangle_A |\Omega\rangle_B |\phi\rangle_{O_1O_2}\\ &+ \frac{b_1^\dagger(x_B) - b_1^\dagger(x_B')}{2}\; a_2^\dagger(x_A)\; |\Omega\rangle_A |\Omega\rangle_B |\phi^\perp\rangle_{O_1O_2}.\end{aligned} \qquad \text{(B4)}$$

For the same reason above, $b_1^\dagger(\cdot)$ commutes with $a_2^\dagger(\cdot)$. Thus we may write

$$\begin{aligned}|\psi\rangle =& \frac{1}{2}\, a_2^\dagger(x_A)\left(b_1^\dagger(x_B) + b_1^\dagger(x_B')\right)\; |\Omega\rangle_A |\Omega\rangle_B |\phi\rangle_{O_1O_2}\\ &+ \frac{1}{2}\, a_2^\dagger(x_A)\left(b_1^\dagger(x_B) - b_1^\dagger(x_B')\right)\; |\Omega\rangle_A |\Omega\rangle_B |\phi^\perp\rangle_{O_1O_2}.\end{aligned} \qquad \text{(B5)}$$

Applying the transformation rule for B,

$$\begin{aligned}b_1^\dagger(x_B) &= b_2^\dagger(x_B)\,|+\rangle\langle+| + b_2^\dagger(x_B')\,|-\rangle\langle-|,\\ b_1^\dagger(x_B') &= b_2^\dagger(x_B)\,|-\rangle\langle-| + b_2^\dagger(x_B')\,|+\rangle\langle+|,\end{aligned} \qquad \text{(B6)}$$

so we have

$$\begin{aligned}b_1^\dagger(x_B) + b_1^\dagger(x_B') &= \left(b_2^\dagger(x_B) + b_2^\dagger(x_B')\right)\left(|+\rangle\langle+| + |-\rangle\langle-|\right)\\ &= \left(b_2^\dagger(x_B) + b_2^\dagger(x_B')\right), \qquad \text{(B7)}\\ b_1^\dagger(x_B) - b_1^\dagger(x_B') &= \left(b_2^\dagger(x_B) - b_2^\dagger(x_B')\right)\left(|+\rangle\langle+| - |-\rangle\langle-|\right), \qquad \text{(B8)}\\ &= \left(b_2^\dagger(x_B) - b_2^\dagger(x_B')\right)\left(|\phi\rangle\langle\phi^\perp| + |\phi^\perp\rangle\langle\phi|\right). \qquad \text{(B9)}\end{aligned}$$

Hence

$$|\psi\rangle = \frac{1}{2}\, a_2^\dagger(x_A)\, \big(b_2^\dagger(x_B) + b_2^\dagger(x'_B)\big)\ |\Omega\rangle_A\, |\Omega\rangle_B\, |\phi\rangle_{O_1O_2} + \frac{1}{2}\, a_2^\dagger(x_A)\, \big(b_2^\dagger(x_B) - b_2^\dagger(x'_B)\big)\ |\Omega\rangle_A\, |\Omega\rangle_B\, |\phi\rangle_{O_1O_2}\,. \tag{B10}$$

$$= a_2^\dagger(x_A)\, b_2^\dagger(x_B)\ |\Omega\rangle_A\, |\Omega\rangle_B\, |\phi\rangle_{O_1O_2}\,.$$

Let us now show that no QC permutation can do this. Assume, for contradiction, that a QC transformation from O_1 to O_2 can map the state to a state in which both particles are created at definite spacetime points. In a QC permutation/diffeomorphism the same control decomposition is used for all particles. Thus there exist orthogonal projectors $\{\pi_k\}$ on $\mathcal{H}_{O_1}\otimes\mathcal{H}_{O_2}$ and bijections $f_k : \mathcal{M} \to \mathbb{R}^4$ such that

$$a_1^\dagger(x) = \sum_k a_2^\dagger\big(f_k(x)\big)\, \pi_k, \quad b_1^\dagger(x) = \sum_k b_2^\dagger\big(f_k(x)\big)\, \pi_k. \tag{B11}$$

Applying (B11) to A in (B2) and bringing $a_2^\dagger$ to the left gives

$$|\psi\rangle = \tag{B12}$$

$$\sum_k \frac{1}{2}\, a_2^\dagger\big(f_k(x_A)\big)\, \big(b_1^\dagger(x_B) + b_1^\dagger(x'_B)\big)\, |\Omega_A\rangle\, |\Omega_B\rangle\, \pi_k\, |\phi\rangle + \sum_k \frac{1}{2}\, a_2^\dagger\big(f_k(x'_A)\big)\, \big(b_1^\dagger(x_B) - b_1^\dagger(x'_B)\big)\, |\Omega_A\rangle\, |\Omega_B\rangle\, \pi_k\, |\phi^\perp\rangle\,. \tag{B13}$$

Applying (B11) to B,

$$|\psi\rangle = \sum_k \Bigg[\frac{1}{2}\, a_2^\dagger\big(f_k(x_A)\big)\, \big(b_2^\dagger(f_k(x_B)) + b_2^\dagger(f_k(x'_B))\big)\ |\Omega_A\rangle\, |\Omega_B\rangle\ \pi_k\, |\phi\rangle + \frac{1}{2}\, a_2^\dagger\big(f_k(x'_A)\big)\, \big(b_2^\dagger(f_k(x_B)) - b_2^\dagger(f_k(x'_B))\big)\ |\Omega_A\rangle\, |\Omega_B\rangle\ \pi_k\, |\phi^\perp\rangle\Bigg]. \tag{B14}$$

If the final state had both particles definite (e.g. proportional to $a_2^\dagger(\tilde{x}_A) b_2^\dagger(\tilde{x}_B)\, |\Omega_A\rangle\, |\Omega_B\rangle\, |\tilde{\phi}\rangle$), then for every k with nonzero amplitude we would need

$$f_k(x_A) = f_k(x'_A) = \tilde{x}_A, \qquad f_k(x_B) = f_k(x'_B) = \tilde{x}_B. \tag{B15}$$

But each f_k is a permutation, so it cannot satisfy this. Therefore no QC permutation can make both particles simultaneously definite for the state $|\psi\rangle$.

Appendix C: Scalar field toy model

In this Appendix we (1) prove the invariance of the action under quantum permutations obeying a differentiability condition (2) present an example of a metric and a transformation that obeying the differentiability condition (3) present a counter-example of a metric and a transformation that do not obey the differentiability condition, and show that in this case the action does not remain invariant.

1. Proof of action invariance

Let us prove something more general than what was stated in the text, namely that the action stays invariant also when the finite difference is divided by the coordinate distance:

$$\Delta\hat{\varphi}_1(x, x_u) := \frac{\hat{\varphi}_1(x_u) - \hat{\varphi}_1(x)}{\|x_u - x\|}, \tag{C1}$$

with $\|\cdot\|$ any norm on $\mathbb{R}^n$. The main text focuses on the case where the norms are absent, or equivalently are all unit. From the differentiability condition, for every pair (x, y) and x_u with $\hat{g}_1(x, x_u, x_v) \neq 0$ at least for one x_v, we have a definite image $D_{xy}(x_u)$. We thus define

$$J_{xy}(x_u) := \frac{\|D_{xy}(x_u) - y\|}{\|x_u - x\|}. \tag{C2}$$

Begin by proving the transformation rule for the finite differences:

$$\Delta\hat{\varphi}_1(x, x_u) = \frac{\hat{\varphi}_1(x_u) - \hat{\varphi}_1(x)}{\|x_u - x\|} = \sum_{y_a} \frac{\big(u_{x_u y_a} - u_{x y_a}\big)\hat{\varphi}_2(y_a)}{\|x_u - x\|}. \tag{C3}$$

Multiplying on the left by u_{xy}, and using $u_{xy}u_{x_u y_a} = 0$ unless $y_a = D_{xy}(x_u)$, together with $u_{xy}u_{xy_a} = 0$ unless $y_a = y$, gives

$$u_{xy}\, \Delta\hat{\varphi}_1(x, x_u) = \frac{u_{xy}u_{x_u\, D_{xy}(x_u)}\hat{\varphi}_2\big(D_{xy}(x_u)\big) - u_{xy}\hat{\varphi}_2(y)}{\|x_u - x\|}. \tag{C4}$$

Using

$$u_{xy}u_{x_u\, D_{xy}(x_u)} = \sum_{y'} u_{xy}u_{x_u y'} = u_{xy}, \tag{C5}$$

we obtain

$$u_{xy}\, \Delta\hat{\varphi}_1(x, x_u) = u_{xy}\, \frac{\hat{\varphi}_2(D_{xy}(x_u)) - \hat{\varphi}_2(y)}{\|x_u - x\|} = u_{xy}\, \frac{\|D_{xy}(x_u) - y\|}{\|x_u - x\|} \Delta\hat{\varphi}_2\big(y, D_{xy}(x_u)\big) = u_{xy}\, J_{xy}(x_u)\, \Delta\hat{\varphi}_2\big(y, D_{xy}(x_u)\big). \tag{C6}$$

Summing over y,

$$\Delta\hat{\varphi}_1(x, x_u) = \sum_y u_{xy}\, J_{xy}(x_u)\, \Delta\hat{\varphi}_2\big(y, D_{xy}(x_u)\big) = \sum_y J_{xy}(x_u)\, \Delta\hat{\varphi}_2\big(y, D_{xy}(x_u)\big)\, u_{xy}. \tag{C7}$$

The second equality follows from repeating the calculation with the QP entries to the right of the field operators.

Now start from

$$S = \sum_{x,x_u,x_v} \Delta\hat{\varphi}_1(x,x_u)\, \hat{g}_1(x,x_u,x_v)\, \Delta\hat{\varphi}_1(x,x_v). \quad \text{(C8)}$$

Substituting (C7),

$$\begin{aligned} S = \sum_{\substack{x,x_u,x_v\\ y,y'}} & J_{xy}(x_u)\, \Delta\hat{\varphi}_2\big(y, D_{xy}(x_u)\big) \\ & u_{xy}\, \hat{g}_1(x,x_u,x_v)\, u_{xy'} \\ & J_{xy'}(x_v)\, \Delta\hat{\varphi}_2\big(y', D_{xy'}(x_v)\big). \end{aligned} \quad \text{(C9)}$$

By assumption the operators $u_{xy}, u_{xy'}$ commute with $\hat{g}_1(x,x_u,x_v)$. Hence for $y \neq y'$,

$$u_{xy}\, \hat{g}_1(x,x_u,x_v)\, u_{xy'} = u_{xy}u_{xy'}\, \hat{g}_1(x,x_u,x_v) = 0, \quad \text{(C10)}$$

since $u_{xy}u_{xy'} = 0$ for $y \neq y'$. Thus,

$$\begin{aligned} S = \sum_{x,x_u,x_v,y} & J_{xy}(x_u)\, \Delta\hat{\varphi}_2\big(y, D_{xy}(x_u)\big) \\ & u_{xy}\, \hat{g}_1(x,x_u,x_v) \\ & J_{xy}(x_v)\, \Delta\hat{\varphi}_2\big(y, D_{xy}(x_v)\big) \\ = \sum_{x,y} \sum_{y_a,y_b} & J_{xy}\big(D_{xy}^{-1}(y_a)\big)\, \Delta\hat{\varphi}_2(y,y_a) \\ & u_{xy}\, \hat{g}_1\big(x, D_{xy}^{-1}(y_a), D_{xy}^{-1}(y_b)\big) \\ & J_{xy}\big(D_{xy}^{-1}(y_b)\big)\, \Delta\hat{\varphi}_2(y,y_b), \end{aligned} \quad \text{(C11)}$$

where in the second equality we changed variables from x_u, x_v to

$$y_a = D_{xy}(x_u), \qquad y_b = D_{xy}(x_v), \quad \text{(C12)}$$

using the invertibility of D_{xy}.

We may therefore define

$$\begin{aligned} \hat{g}_2(y,y_a,y_b) := \sum_x u_{xy}\, J_{xy}\big(D_{xy}^{-1}(y_a)\big)\, J_{xy}\big(D_{xy}^{-1}(y_b)\big) \\ \hat{g}_1\big(x, D_{xy}^{-1}(y_a), D_{xy}^{-1}(y_b)\big), \end{aligned} \quad \text{(C13)}$$

so that

$$S = \sum_{y,y_a,y_b} \Delta\hat{\varphi}_2(y,y_a)\, \hat{g}_2(y,y_a,y_b)\, \Delta\hat{\varphi}_2(y,y_b). \quad \text{(C14)}$$

Thus the action retains its form, with the derivative factors $J_{xy}(x_u)$ entering the transformation laws in an analogous manner to GR.

Moreover, since

$$\hat{g}_1(x,x_u,x_v) = \hat{g}_1(x,x_v,x_u) = \hat{g}_1(x,x_u,x_v)^\dagger, \quad \text{(C15)}$$

the same holds for $\hat{g}_2$. Indeed,

$$\begin{aligned} \hat{g}_2(y,y_a,y_b)^\dagger &= \sum_x u_{xy}\, J_{xy}\big(D_{xy}^{-1}(y_a)\big)\, J_{xy}\big(D_{xy}^{-1}(y_b)\big) \\ &\qquad \hat{g}_1\big(x, D_{xy}^{-1}(y_a), D_{xy}^{-1}(y_b)\big)^\dagger \\ &= \sum_x u_{xy}\, J_{xy}\big(D_{xy}^{-1}(y_a)\big)\, J_{xy}\big(D_{xy}^{-1}(y_b)\big) \\ &\qquad \hat{g}_1\big(x, D_{xy}^{-1}(y_a), D_{xy}^{-1}(y_b)\big) \\ &= \hat{g}_2(y,y_a,y_b), \end{aligned} \quad \text{(C16)}$$

and since $\hat{g}_1$ is symmetric in its last two arguments, so is $\hat{g}_2$. Hence

$$\hat{g}_2(y,y_a,y_b) = \hat{g}_2(y,y_a,y_b)^\dagger = \hat{g}_2(y,y_b,y_a). \quad \text{(C17)}$$

Commutation of $\hat{g}_2(y,y_a,y_b)$ with u_{xy} for all x also follows immediately from the definition.

2. Concrete example

A concrete example for the differentiability condition is obtained from four disjoint isomorphic copies of the same graph Λ comprising together a graph Γ. Denote the corresponding vertices by $1_i, 2_i, 3_i, 4_i$, where i runs over the vertices of Λ, and take any $\hat{g}_1(x,x_u,x_v)$ that is zero unless $x \sim_\Gamma x_u, x_v$. Define the following BQC:

$$\begin{aligned} u_{1_i\,1_i} &= \pi, & u_{1_i\,2_i} &= \mathbb{1} - \pi, \\ u_{2_i\,1_i} &= \mathbb{1} - \pi, & u_{2_i\,2_i} &= \pi, \\ u_{3_i\,3_i} &= \pi', & u_{3_i\,4_i} &= \mathbb{1} - \pi', \\ u_{4_i\,3_i} &= \mathbb{1} - \pi', & u_{4_i\,4_i} &= \pi', \end{aligned} \quad \text{(C18)}$$

for every i, with all other entries equal to zero, where π, π' are two non-commuting orthogonal projections on $\mathcal{H}_{O_1} \otimes \mathcal{H}_{O_2}$.

This u is differentiable. Indeed, if $1_j \sim_\Gamma 1_i$, then

$$D_{1_i\,1_i}(1_j) = 1_j, \qquad D_{1_i\,2_i}(1_j) = 2_j, \quad \text{(C19)}$$

and similarly, if $2_j \sim_\Gamma 2_i$,

$$D_{2_i\,2_i}(2_j) = 2_j, \qquad D_{2_i\,1_i}(2_j) = 1_j. \quad \text{(C20)}$$

Likewise, for $3_j \sim_\Gamma 3_i$ and $4_j \sim_\Gamma 4_i$,

$$\begin{aligned} D_{3_i\,3_i}(3_j) &= 3_j, & D_{3_i\,4_i}(3_j) &= 4_j, \\ D_{4_i\,4_i}(4_j) &= 4_j, & D_{4_i\,3_i}(4_j) &= 3_j. \end{aligned} \quad \text{(C21)}$$

3. Counter example: a square

An example of a QP that is not differentiable and does not preserve the action above is the following QP on 4 elements:

$$\begin{aligned} &u_{11} = \pi, \qquad u_{12} = \mathbb{1} - \pi, \\ &u_{21} = \mathbb{1} - \pi, \qquad u_{22} = \pi, \\ &u_{33} = \pi', \qquad u_{34} = \mathbb{1} - \pi', \\ &u_{43} = \mathbb{1} - \pi', \qquad u_{44} = \pi', \end{aligned} \tag{C22}$$

with all other entries equal to zero, where again $\hat{g}_1(x, x_u, x_v) = 0$ unless $x \sim_\Gamma x_u, x_v$, with Γ the square graph.

Indeed, since $3 \sim_\Gamma 1$, take $x = 1$, $x_u = 3$ and $y = 1$. One has

$$u_{11}u_{33} = \pi\pi' \neq 0, \qquad u_{11}u_{34} = \pi(\mathbb{1} - \pi') \neq 0. \tag{C23}$$

Thus there are two distinct values, $y_a = 3$ and $y_a = 4$, such that $u_{11}u_{3y_a} \neq 0$, so no definite value $D_{11}(3)$ exists. Hence u is not differentiable.

To see explicitly that the action is not preserved, it is enough to choose a particularly simple $\hat{g}_1$. Let

$$\hat{g}_1(1,3,3) = \hat{G} \neq 0, \tag{C24}$$

and all other $\hat{g}_1(x, x_u, x_v)$ vanish. Then in the O_1 reference the action is simply

$$S = \Delta\hat{\varphi}_1(1,3)\, \hat{G}\, \Delta\hat{\varphi}_1(1,3). \tag{C25}$$

Using the transformation rule for the field,

$$\hat{\varphi}_1(x) = \sum_y u_{xy}\, \hat{\varphi}_2(y) = \sum_y \hat{\varphi}_2(y)\, u_{xy}, \tag{C26}$$

we have

$$\begin{aligned} \hat{\varphi}_1(1) &= \pi\, \hat{\varphi}_2(1) + (\mathbb{1} - \pi)\, \hat{\varphi}_2(2), \\ \hat{\varphi}_1(3) &= \pi'\, \hat{\varphi}_2(3) + (\mathbb{1} - \pi')\, \hat{\varphi}_2(4). \end{aligned} \tag{C27}$$

Hence

$$\begin{aligned} \|3-1\| \Delta\hat{\varphi}_1(1,3) =& \hat{\varphi}_1(3) - \hat{\varphi}_1(1) \\ =& \pi'\hat{\varphi}_2(3) + (\mathbb{1} - \pi')\hat{\varphi}_2(4) \\ & - \pi\hat{\varphi}_2(1) - (\mathbb{1} - \pi)\hat{\varphi}_2(2). \end{aligned} \tag{C28}$$

Substituting this, together with the corresponding expansion with the field operators to the left of the QP entries, gives

$$\begin{aligned} S = \frac{1}{4}\Big(& \hat{\varphi}_2(3)\pi' + \hat{\varphi}_2(4)(\mathbb{1} - \pi') \\ & - \hat{\varphi}_2(1)\pi - \hat{\varphi}_2(2)(\mathbb{1} - \pi)\Big) \\ & \hat{G} \\ \Big(& \pi'\hat{\varphi}_2(3) + (\mathbb{1} - \pi')\hat{\varphi}_2(4) \\ & - \pi\hat{\varphi}_2(1) - (\mathbb{1} - \pi)\hat{\varphi}_2(2)\Big). \end{aligned} \tag{C29}$$

This can be written as

$$S = \sum_{i,j=1}^{4} \hat{\varphi}_2(i)\, \hat{A}_{ij}\, \hat{\varphi}_2(j), \tag{C30}$$

with for example

$$\hat{A}_{31} = -\frac{1}{4}\pi'\hat{G}\pi \neq \hat{A}_{31}^\dagger. \tag{C31}$$

This cannot come from an action of the form

$$\sum_{y, y_a, y_b} \Delta\hat{\varphi}_2(y, y_a)\, \hat{g}_2(y, y_a, y_b)\, \Delta\hat{\varphi}_2(y, y_b), \tag{C32}$$

with

$$\hat{g}_2(y, y_a, y_b) = \hat{g}_2(y, y_a, y_b)^\dagger = \hat{g}_2(y, y_b, y_a), \tag{C33}$$

since expanding this form yields $\hat{A}_{ij}$ that are a linear combination of different $\hat{g}_2(y, y_a, y_b)$ with real coefficients ± 1. Therefore, such a form necessarily yields $\hat{A}_{ij} = \hat{A}_{ij}^\dagger$ for all i, j.

Appendix D: Proof that a quantum permutation is quantum controlled if and only if all of its entries commute

Let $u = (u_{xy})_{1 \leq x, y \leq N}$ be a 'classical' quantum permutation, i.e. all u_{xy} commute with each other. The proof of Lemma 2.4 in [19] constructs for such a u a quantum permutation $w^{(a)}$ and a unitary W_a on $\mathcal{H}_O$ (which they denote as $\mathcal{H}$) such that $w_{xy}^{(a+1)} = W_{a+1} w_{xy}^{(a)} W_{a+1}^\dagger$ and $w_{xy}^{(1)} = W_1 u_{xy} W_1^\dagger$ for all x, y and $a = 1, ..., N$. Moreover, their construction ensures that for the last quantum permutation in this sequence, $w := w^{(N)}$, its entries w_{xy} are all diagonal and containing only 0 and 1.

Let us define now the unitary $W := W_N W_{N-1} \cdots W_1$. It follows from the above that $w_{xy} = W u_{xy} W^\dagger$ for all x, y. Therefore,

$$\begin{aligned} u_{xy} &= W^\dagger w_{xy} W = W^\dagger \sum_{k=1}^{d} (w_{xy})_{kk} \left|k\right\rangle \left\langle k\right| \;\; W \\ &= \sum_k (\sigma_k)_{xy}\, \pi_k = \left(\sum_k \sigma_k \otimes \pi_k\right)_{xy}, \end{aligned} \tag{D1}$$

where we defined operators $\pi_k := W^\dagger \left|k\right\rangle \left\langle k\right| W$ on $\mathcal{H}_O$ and σ_k such that $(\sigma_k)_{xy} = (w_{xy})_{kk}$. From w being a quantum permutation and from w_{xy} being diagonal and containing only 0 and 1, it follows that σ_k are permutation matrices on N elements. Moreover, it is easy to see that π_k are orthogonal projections on $\mathcal{H}_O$ that sum to the identity. Therefore, u fulfills the definition of a QC permutation.

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